Correlation functions in Minimal Liouville Gravity from Douglas string equation

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Introduction

Minimal Liouville gravity (MLG) was invented and partly solved by Knizhnik , Polyakov and A. Zamolodchikov in 1987 . I will use David-Distler-Kawai formulation of MLG.

It is an example of Liouville gravity, matter sector is (q, p) Minimal Model of CFT .

The N-point sphere correlation functions given by the integrals over the moduli space $M_{0,N}$ of Riemann surfaces of genus 0 with N punctures:

$$Z_{m_1 n_1 \ldots m_N n_N} = \int_{\mathcal{M}_{0,N}} \langle \mathcal{O}_{m_1 n_1} \ldots \mathcal{O}_{m_N n_N} \rangle dz_1 d\bar{z}_1 dz_2 d\bar{z}_2 \ldots dz_{N-3} d\bar{z_{N-3}}$$

where $z_1 \ldots z_{N-3}$ are coordinates on $\mathcal{M}_{0,N}$, $O_{mn} = \int_X \mathcal{O}_{m,n} dz d\bar{z}$, X — 2d surface, $\mathcal{O}_{m,n}$ — is primary field , which is $(1,1)$ form on X .

The important progress in computation the integrals over the moduli space M was achieved using Liouville Higher Equations of motion of Aleosha Zamolodchikov.

It is convenient to define the generating function

$$Z_L(\lambda) = \langle \exp \sum_{m,n} \lambda_{m,n} O_{m,n} \rangle,$$

$$Z_{m_1 n_1 \ldots m_N n_N} = \frac{\partial}{\partial \lambda_{m_1 n_1}} \ldots \frac{\partial}{\partial \lambda_{m_N n_N}} \bigg|_{\lambda=0} Z_L(\lambda)$$

$Z_L(\lambda)$ can be considered as the partition function of the perturbed MLG .
The coupling constants $\lambda_{m,n}$ are the coordinates on the space of the perturbed MLG .

The another discrete approach to 2-dimensional gravity was invented and solved (Kazakov , Kostov,Migdal, Brezin ,Douglas ,Shenker, Gross ,Staudacher) .

In this approach , realized through Matrix Models , the fluctuating 2-dimensional surfaces are approximated by an ensemble of graphs. The continuous geometry is restored in the scaling limit, when large size graphs dominate.

These two approaches , based on the same idea of 2-dimensional fluctuating geometry, had been expected to give the same results .

In 1990 Douglas discovered that the partition function in the discrete approach satisfies the KdV and so-called "string equation" .

The KdV times $\tau_{m,n}$ of the generalized KdV hierarchy play a role of the perturbation parameters $\lambda_{m,n}$ in Minimal Liouville gravity. Their gravitational dimensions coincide .

However a direct identification of the times $\tau_{m,n}$ and coupling constants $\lambda_{m,n}$ leads to inconsistencies.

In Minimal Gravity one has conformal and fusion selection rules. For instance, the one-point correlation numbers of all operators must be zero, the two-point correlation numbers must be diagonal .

There exist similar restrictions to higher point correlation numbers. They are not satisfied in the Douglas approach if we identify $\tau_{m,n}$ and $\lambda_{m,n}$.

This problem was pointed out and partly solved by A.Zamolodchikov , Moore, Seiberg and Staudacher .

The idea was due to possible contact terms in OPE the times $\tau_{m,n}$ in the Douglas approach and coupling constants $\lambda_{m,n}$ in Minimal Liouville Gravity are related in a non-linear fashion

$$\tau_{m,n} = \lambda_{m,n} + \sum_{m_1 n_1 m_2 n_2} C_{m,n}^{m_1 n_1 m_2 n_2} \lambda_{m_1 n_1} \lambda_{m_2 n_2} + \dots$$

and an appropriate choice of such substitution allowed them to reach the coincidence up to 2p correlation numbers in $(2, 2s+1)$ Minimal Gravity.

After the 4-point correlation numbers in Minimal Gravity had been calculated , it became possible to make the new explicit checks against Douglas equation approach.

It was performed for $(2, 2s+1)$ Minimal Gravity and the full correspondence between Douglas equation approach and Minimal Liouville gravity was reached .

The our aim is to generalize the results for $(2, 2s+1)$ case to the case of $(3, 3s+1)$ and $(3, 3s+2)$ Minimal Gravity.

Our analysis is based on the following assumptions:

• There exist special coordinates $\tau_{m,n}$ in the space of perturbed Minimal Liouville Gravities such that the partition function of Minimal Liouville Gravity satisfies the Douglas string equation.

• String equation define the partition function as logarithm of Sato's tau-function of the dispersionless generalized KdV hierarchy with the initial conditions defined by the Douglas string equation.

• The times $\tau_{m,n}$ are related to the natural coordinates $\lambda_{m,n}$ on the space of the perturbed Minimal Liouville Gravities by a non-linear transformation like.

• We establish the form of this transformation $\tau(\lambda)$ by the requirement that in the correlation numbers, which are the coefficients of the expansion of the partition function in the coordinates $\lambda_{m,n}$, satisfy the conformal and fusion selection rules.

• To perform this we we get a convenient expression for the partition function using the relation of String equation with a Frobenius manifold structure .

The Minimal Liouville gravity consists of Liouville theory of field ϕ and some matter sector which is taken to be a (q, p) Minimal Model of CFT .

The Minimal Model $\mathcal{M}_{q,p}$ has primary fields, which are enumerated by the Kac table: $\Phi_{m,n}$, where $m = 1, \ldots, q-1$ and $n = 1, \ldots, p-1$.
Only a half of the fields $\Phi_{m,n}$ are independent.

$$\Phi_{m,n} = \Phi_{q-m,p-n}$$

The operator product expansion (OPE) for these fields is

$$[\Phi_{m_1,n_1}][\Phi_{m_2,n_2}] = \sum_{m=|m_1-m_2|:2}^{I(m_1,m_2)} \sum_{n=|n_1-n_2|:2}^{I(n_1,n_2)} [\Phi_{m,n}]$$

$[\Phi_{m,n}]$ denotes the contribution of the irreducible Virasoro representation with the highest state $\Phi_{m,n}$.
Summation goes with the step 2 and

$$I(m_1, m_2) = \min(m_1 + m_2 - 1, 2q - m_1 - m_2 - 1).$$

$$I(m_1, m_2) = \min(m_1 + m_2 - 1, 2q - m_1 - m_2 - 1).$$

The small conformal group and OPE put strong constraints on correlation functions.For instance constraints for one- and two- point correlation functions

$$\langle \Phi_{m,n}(x) \rangle = 0,$$

$$\langle \Phi_{m_1,n_1}(x_1)\Phi_{m_2,n_2}(x_2) \rangle = 0, \qquad m_1, n_1 \neq m_2, n_2.$$

For higher correlation numbers we use the OPE fusion ,then the conformal rules.

For instance the three-point correlation functions satisfy

$$\langle \Phi_{m_1,n_1}\Phi_{m_2,n_2}\Phi_{m_3,n_3} \rangle = 0, \quad m_3 > l(m_1, m_2) = \begin{cases} m_1 + m_2 - 1, & m_1 + m_2 - 1 \leq q - 1 \\ 2q - m_1 - m_2 - 1, & m_1 + m_2 > q \end{cases}$$

where we assume that $m_1 \leq m_2 \leq m_3$.

These and similar equations we call *selection rules*.

The Polyakov's continuous approach to two-dimensional quantum gravity is defined through the path integral over two-dimensional Riemannian metrics $g_{\mu\nu}$ interacting with some conformal matter.

In conformal gauge $g_{\mu\nu} = e^{\phi}\hat{g}_{\mu\nu}$ it leads to Liouville action

$$S_L = \frac{1}{4\pi}\int_M \sqrt{\hat{g}}\left(\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + Q\hat{R}\phi + 4\pi\mu e^{2b\phi}\right) d^2x$$

where $\hat{g}_{\mu\nu}$ is some fixed background metric, μ – cosmological constant and parameters Q, b are related to the central charge c_L of the Liouville theory

$$c_L = 1 + 6Q^2, \qquad Q = b + b^{-1}.$$

The central charge c_M of the conformal matter is related to the central charge of the Liouville theory by the Weyl anomaly cancellation condition

$$c_L + c_M = 26.$$

In the case of (q,p) Minimal Liouville Gravity where the conformal matter is (q,p) Minimal Model of CFT, we have $b = \sqrt{\frac{q}{p}}$.

Correlation numbers of MLG

The observables of the (q, p) are cohomologies of an appropriate BRST operator.

They are enumerated as primary fields in corresponding Minimal Model and denoted as $O_{m,n}$. Explicitly

$$O_{m,n} = \int_{x \in M} \mathcal{O}_{m,n}(x), \qquad \mathcal{O}_{m,n}(x) = \Phi_{m,n}(x) e^{2b\delta_{m,n}\phi(x)} \sqrt{\hat{g}} d^2 x$$

The operators $O_{m,n}$ satisfy the same selection rules as $\Phi_{m,n}$ and the scaling property

$$O_{m,n} \sim \mu^{-\delta_{m,n}}, \qquad \delta_{m,n} = \frac{p + q - |pm - qn|}{2q}.$$

We define in Minimal Liouville Gravity correlation numbers as

$$Z_{m_1 n_1 \dots m_N n_N} = \langle O_{m_1,n_1} \dots O_{m_N,n_N} \rangle$$

The generating function of these correlation numbers

$$Z_L(\{\lambda_{m,n}\}) = \left\langle \exp \sum_{m,n} \lambda_{m,n} O_{m,n} \right\rangle.$$

is a quasihomogeneous function

$$Z_L(\{\rho^{\delta_{m,n}} \lambda_{m,n}\}) = \rho^{\frac{p+q}{q}} Z_L(\{\lambda_{m,n}\})$$

3- and 4-point functions of MLG

The explicite formulae for two-, three- and four-point correlation numbers in MLG are known from direct computations .

We can compare these results with those from the Douglas string equation.

To make sensible comparisons we write down the quantities which do not depend on the normalizations of operators and correlators

$$\frac{\langle\langle O_{m_1,n_1} O_{m_2,n_2} O_{m_3,n_3}\rangle\rangle^2}{\prod_{i=1}^{3}\langle\langle O^2_{m_i,n_i}\rangle\rangle} = \frac{\prod_{i=1}^{3}|m_i p - n_i q|}{p(p+q)(p-q)}$$

$$\frac{\langle\langle O_{m_1,n_1} O_{m_2,n_2} O_{m_3,n_3} O_{m_4,n_4}\rangle\rangle}{\left(\prod_{i=1}^{4}\langle\langle O^2_{m_i,n_i}\rangle\rangle\right)^{\frac{1}{2}}} =$$

$$= \frac{\prod_{i=1}^{4}|m_i p - n_i q|}{2p(p+q)(p-q)}\left(\sum_{i=2}^{4}\sum_{r=-(m_1-1)}^{m_1-1}\sum_{t=-(n_1-1)}^{n_1-1}|(m_i-r)p-(n_i-t)q| - m_1 n_1(m_1 p + n_1 q)\right)$$

where $\langle\langle\ldots\rangle\rangle = \frac{\langle\ldots\rangle}{\langle 1\rangle}$. Sums over r, t in the last formula are with the step 2.

Contact terms

The correlation numbers involve integration over n points on the $2d$ surface M

$$Z_{m_1 n_1 \dots m_N n_N} = \int \langle \mathcal{O}_{m_1,n_1}(x_1) \dots \mathcal{O}_{m_N,n_N}(x_N) \rangle d^2 x_1 \dots d^2 x_{N-3}$$

There could get contact delta-like terms when two or more points x_i are coincident.

The ambiguity in contact terms leads to the fact that we can add to the n-point correlation numbers some k-point correlation numbers. Like this

$$\langle O_{m_1,n_1} O_{m_2,n_2} \rangle \to \langle O_{m_1,n_1} O_{m_2,n_2} \rangle + \sum_{m,n} A_{m,n}^{(m_1 n_1)(m_2 n_2)} \langle O_{m,n} \rangle.$$

Such a substitution is equivalent to a change of coupling constants in the generating function

$$\lambda_{m,n} \to \lambda_{m,n} + \sum_{m_1,n_1,m_2,n_2} A_{m,n}^{(m_1 n_1)(m_2 n_2)} \lambda_{m_1 n_1} \lambda_{m_2 n_2}.$$

In MLG we can impose a restriction on this change of coupling constants because they have certain scaling dimension

$$\lambda_{m,n} \sim \mu^{\delta_{m,n}}$$

Therefore we can demand all the terms to have the same dimensions as one of $\lambda_{m,n}$

$$\delta_{m,n} = \delta_{m_1,n_1} + \delta_{m_2,n_2} + \delta_{m_3,n_3} + \dots$$

However in the Liouville Minimal Gravity such resonance cases are very common and a freedom of the change still remains .

Contact terms

Any addition of contact terms of such kind is equivalent
to some non-linear polynomial change of coupling constants

$$\lambda_{m,n} \to A\mu^{\delta_{m,n}} + \sum_{m_1,n_1} C_{m,n}^{(m_1 n_1)} \mu^{\delta_{m,n}-\delta_{m_1,n_1}} \lambda_{m_1,n_1}$$
$$+ \sum_{m_1,n_1} \sum_{m_2,n_2} C_{m,n}^{(m_1 n_1)(m_2 n_2)} \mu^{\delta_{m,n}-\delta_{m_1,n_1}-\delta_{m_2,n_2}} \lambda_{m_1 n_1} \lambda_{m_2 n_2} + \ldots .$$

Only the terms in these sums with the integer and positive degrees

$$\delta_{m,n} - \delta_{m_1,n_1} - \delta_{m_2,n_2} - \ldots .$$

have nonvanishing coefficients .

So, there exist different "systems of coordinates" of $\lambda_{m,n}$ and in general case MLG coordinate frame and the natural one for Douglas string equation may do not coincide .

This change of variables conserves the quasi-homogeneity property

$$Z_L(\{\rho^{\delta_{m,n}} \lambda_{m,n}\}) = \rho^{\frac{p+q}{q}} Z_L(\{\lambda_{m,n}\})$$

Douglas string equation

Due to Douglas the partition function of (q,p) Minimal Gravity, is described as follows. One takes differential operators P and Q of the form

$$Q = d^q + \sum_{\alpha=1}^{q-1} u_\alpha(x) d^{q-\alpha-1},$$

$$P = \left(Q^{\frac{p}{q}} + \sum_{\alpha=1}^{q-1} \sum_{k=1} \tilde{t}_{k,\alpha} Q^{k+\frac{\alpha}{q}-1} \right)_+$$

where $d = \frac{d}{dx}$, $Q^{\frac{a}{q}}$ is the pseudo-differential operators and $(\dots)_+$ means its non-negative part . The constants $\tilde{t}_{k,\alpha}$ are KdV times. Then the so-called *String equation* looks as

$$[P, Q] = 1$$

The free energy $\mathcal{F}(\tilde{t})$ is defined from the equation

$$\frac{\partial^2 \mathcal{F}}{\partial x^2} = u_1^*$$

where u_α^* is an appropriate solution of the string equation .

In the genus 0 case $\frac{d}{dx}$ is replaced with a variable y and the commutator in string equation is replaced with Poisson bracket

$$[P, Q] = 1 \to \{P, Q\} = \frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Q}{\partial x} = 1.$$

In the quasiclassical limit we take Q and P to be just polynomials of y

$$Q = y^q + \sum_{\alpha=1}^{q-1} u_\alpha y^{q-\alpha-1}$$

String equation is equivalent to Principle of least action :

$$\frac{\partial S}{\partial u^\alpha} = 0$$

where

$$S = Res\left(Q^{\frac{p}{q}+1} + \sum_{\alpha=1}^{q-1}\sum_{k=0} t_{k,\alpha} Q^{k+\frac{\alpha}{q}}\right) = S_{s+1,p_0} + \sum_{\alpha=1}^{q-1}\sum_{k=0} t_{k,\alpha} S_{k,\alpha}$$

where $(q,p) = (q, sq+p_0)$, $1 \leq p_0 \leq q-1$, $t_{0,1} = x$

$$S_{k,\alpha} = Res(Q^{k+\frac{\alpha}{q}})$$

and the residue is taken at $y = \infty$.

To identify the free energy of String equation and the partition function of the Liouville theory , we first perform the dimension analysis.

$$Z_L \sim \mu^{\frac{p+q}{q}}$$

where μ is the cosmological constant.

On the other hand the equation $\frac{\partial^2 \mathcal{F}}{\partial x^2} = u_*^1$ gives that

$$Z \sim x^2 u^1 \sim y^{2(p+q)}$$

where the scaling dimension of x is determined from the equation $\{P, Q\} = 1$ and scaling dimension of u_α from the demand that all terms in Q are of the same dimension

$$x \sim y^{p+q-1}, \qquad u_\alpha \sim y^{\alpha+1}$$

Thus if we want $\mathcal{F} \sim Z_L$ we get

$$y \sim \mu^{\frac{1}{2q}}$$

This follows

$$t_{k,\alpha} \sim \mu^{\frac{s+1-k}{2}+\frac{p_0-\alpha}{2q}}.$$

In particular $t_{s-1,p_0} \sim \mu$.

To identify these times and their dimensions with the dimensions of the coupling constants $\lambda_{m,n}$ in the MLG whose scaling dimensions are

$$\delta_{m,n} = \frac{p+q-|pm-qn|}{2q}.$$

we write it in the form $S = Res\left(Q^{\frac{p+q}{q}} + \sum_{m,n}\tau_{m,n}Q^{|pm-qn|}\right)$.

Then the dimension of times $\tau_{m,n}$ is (since $Q \sim y^q \sim \mu^{\frac{1}{2}}$)

$$\tau_{m,n} \sim \mu^{\frac{1}{2}\left(\frac{p+q}{q}-|pm-qn|\right)} \sim \lambda_{m,n}$$

Transformation from KdV-frame to MLG-frame

So we get the expression the action, in which the correspondence with the operator from MLG is evident

$$S = Res\left(Q^{\frac{p+q}{q}} + \sum_{m=1}^{q-1}\sum_{n=1}^{p-1}\tau_{m,n}Q^{\frac{|pm-qn|}{q}}\right).$$

Thus the dimensions of times $\tau_{m,n}$ coincide with the dimensions of $\lambda_{m,n}$.

But these two sets of variables do not necessarily coincide they can have a non-linear relation like

$$\tau_{m,n} = A\mu^{\delta_{m,n}} + \sum_{m_1,n_1} C_{m,n}^{(m_1 n_1)}\mu^{\delta_{m,n}-\delta_{m_1,n_1}}\lambda_{m_1,n_1} + \sum_{m_1,n_1}\sum_{m_2,n_2} C_{m,n}^{(m_1 n_1)(m_2 n_2)}\mu^{\delta_{m,n}-\delta_{m_1,n_1}-\delta_{m_2,n_2}}\lambda_{m_1 n_1}\lambda_{m_2 n_2} + \dots.$$

These relations are found from requirement of the selection rules.

For $(2, 2s+1)$ Minimal Gravities it was done a few years ago (AB,BZ) .
Recently a progess for cases of $(3, 3s+1)$, $(3, 3s+2)$ was reached .

String equation and Frobenius manifold structure

The free energy $\mathcal{F}$ in the Douglas approach is related with the suitable solution of String equation u_1^* as

$$\frac{\partial^2 \mathcal{F}}{\partial x^2} = u_*^1$$

To get us a more explicite expression for $\mathcal{F}$ we use the connection between String equation and Frobenius manifold structure .

Let $\mathcal{A}$ is an algebra of polynomials mod Q', where prime is the derivative over y

$$\mathcal{A} = C[y]/Q'.$$

Also define a bilinear form on the space of such polynomials

$$(P_1(y), P_2(y)) := Res\left(\frac{P_1(y)P_2(y)}{Q'}\right),$$

the residue is taken at $y = \infty$. Introducing some basis ϕ_α of $\mathcal{A}$

$$\phi_\alpha \phi_\beta = C_{\alpha\beta}^{\gamma} \phi_\gamma \quad \text{mod } (Q').$$

One then have that

$$Res \frac{\phi_\alpha \phi_\beta \phi_\gamma}{Q'} = C_{\alpha\beta}^{\delta} \cdot Res \frac{\phi_\delta \phi_\gamma}{Q'} = C_{\alpha\beta}^{\delta} g_{\delta\gamma} = C_{\alpha\beta\gamma}$$

where the indices are raised and lowered by the metric $g_{\alpha\beta} = (\phi_\alpha, \phi_\beta)$

There are two useful bases ϕ_α. The first one is of the monomials y^α ,another one

$$\phi_\alpha = \frac{\partial Q}{\partial v^\alpha}$$

where $v_\alpha = -\frac{q}{q-\alpha} Res Q^{\frac{q-\alpha}{q}}$ are flat coordinates on M , in these coordinates $g_{\alpha\beta} = \delta_{\alpha+\beta,q}$ and

$$C_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\gamma}.$$

the structure constants $C_{\alpha\beta\gamma}$ are evaluated in the bases $\frac{\partial Q}{\partial v^\alpha}$. These formulae provide Frobenius manifold structure for $\mathcal{A}$ case. We show

$$\mathcal{F} = \frac{1}{2}\int_0^{u_*} C_\alpha^{\beta\gamma} \frac{\partial S}{\partial u^\beta} \frac{\partial S}{\partial u^\gamma} du^\alpha$$

To prove this we need to verify two points.
The first is that one-form $\Omega = C_\alpha^{\beta\gamma} \frac{\partial S}{\partial v^\beta} \frac{\partial S}{\partial v^\gamma} dv^\alpha$ is closed.
It is verified by taking de Rham differential of Ω and using the associativity of $\mathcal{A}$ together the recursion relation for $S_{k,\alpha}$

$$C_{\alpha\beta}^\gamma C_{\gamma\delta}^\phi = C_{\alpha\gamma}^\phi C_{\beta\delta}^\gamma,$$

$$\frac{\partial^2 S_{n,\alpha}}{\partial v^\beta \partial v^\gamma} = C_{\beta\gamma}^\delta \frac{\partial S_{n-1,\alpha}}{\partial v^\delta}$$

The second point is that $\frac{\partial^2 \mathcal{F}}{\partial x^2} = u_1^*$. This is verified by direct differentiation $\mathcal{F}$ and using that $t_{0,1} = x$ and $C_{\alpha 1}^\beta = \delta_\alpha^\beta$.

Having the explicit expression for the partition function
we calculate the correlation numbers in two steps.

We insert the resonance substitution $t(\lambda)$ in the partition function
and then take derivatives for $\lambda_{m,n} = 0$ except for the cosmological constant $\lambda_{1,1} = \mu$.

$$Z_{m_1 n_1 \ldots m_N n_N} = \frac{\partial}{\partial \lambda_{m_1 n_1}} \cdots \frac{\partial}{\partial \lambda_{m_N n_N}} \Bigg|_{\substack{\lambda_{m,n}=0 \\ \text{for } (m,n)\neq(1,1)}} \mathcal{F}[t(\lambda)].$$

Firstly we are going to review as a simple example $(2, 2s+1)$ Minimal Liouville gravity (A.Zamolodchikov and A.B.).
In this case we have the polynomial Q and the free energy

$$Q = y^2 + u, \qquad \mathcal{F} = \frac{1}{2}\int_0^{u_*} S_u^2(u)du$$

where u_* is the solution of the equation

$$S_u \equiv \frac{\partial S}{\partial u} = u^{s+1} + \sum_{n=1}^{s} \tau_{1,n} u^{s-n} = 0$$

we have changed the normalization of the times $\tau_{m,n}$.

$$S_u = u^{s+1} + t_0 u^{s-1} + \sum_{k=1}^{s-1} t_k u^{s-k-1}$$

where $t_0 = \tau_{1,1} \sim \mu$ and we redenote $t_n = \tau_{1,n+1}$.

To get the generating function for the correlation numbers one inserts the resonance relations

$$t_k = \lambda_k + \sum C_k^{k_1\dots k_n} \lambda_{k_1} \dots \lambda_{k_n}$$

into the partition function and polynomial S_u. The result is of the form

$$\mathcal{F} = Z_0 + \sum_{k=1}^{s-1} \lambda_k Z_k + \frac{1}{2} \sum_{k_1,k_2=1}^{s-1} \lambda_{k_1}\lambda_{k_2} Z_{k_1 k_2} + \dots$$

$$S_u = S_u^0 + \sum_{k=1}^{s-1} \lambda_k S_u^k + \frac{1}{2} \sum_{k_1,k_2=1}^{s-1} \lambda_{k_1}\lambda_{k_2} S_u^{k_1 k_2} + \dots$$

Also from the original form of the polynomial S_u one finds that

$$S_u^0(u) = u^{s+1} + B\mu u^{s-1} + C\mu^2 u^{s-3} + \dots$$

$$S_u^k(u) = A_k u^{s-k-1} + B_k \mu u^{s-k-3} + C_k \mu^2 u^{s-k-5} + \dots$$

$$S_u^{k_1 k_2}(u) = A_{k_1 k_2} u^{s-k_1-k_2-3} + B_{k_1 k_2} \mu u^{s-k_1-k_2-5} + C_{k_1 k_2} \mu^2 u^{s-k_1-k_2-k_3-7} + \dots$$

where all the polynomials have certain parity since $\mu \sim u^2$.

As above we find that the dimensions in this case are

$$\lambda_k \sim \mu^{\frac{k+2}{2}}$$

$$\mathcal{F} \sim \mu^{\frac{2s+3}{2}}$$

$$Z_{k_1 \dots k_n} \sim \mu^{\frac{2s+3-\sum(k_i+2)}{2}}$$

As usually in the spirit of the scaling theory of criticality, we are interested only in the singular part of the partition function and disregard the regular part as non-universal.

Notice that $Z_{k_1 \dots k_n}$ is always singular if $\sum k_i$ is even.

On the other hand when $\sum k_i$ is odd and additionally

$$\sum_{i=1}^{n} k_i \leq 2s + 3 - 2n$$

the correlation number $Z_{k_1 \dots k_n}$ involves only non-negative integer powers of μ and thus is non-singular.

This inequality always holds for one- and two-point correlation numbers. So we shall consider the sector of odd $\sum k_i$ only starting from the three point correlation numbers.

It is convenient to switch to dimensionless quantities

$$s_k = \frac{g_k}{g} u_0^{-(k+2)} \lambda_k$$
$$S_u(u) = g u_0^{s+1} Y_u(u/u_0)$$
$$\mathcal{F} = g^2 u_0^{2s+3} \mathcal{Z}$$

where $u_0 = u_*(\lambda = 0) \sim \mu^{\frac{1}{2}}$, $g_k = \frac{(p-k-1)!}{(2p-2k-3)!!}$ and $g = \frac{(p+1)!}{(2p+1)!!}$
Then one has

$$\mathcal{Z} = \frac{1}{2} \int_0^{x_*} Y_u^2(x) dx$$

where $x = \frac{u}{u_0}$ and $x_* = x_*(s)$ is an appropriate zero of the polynomial $Y_u(x)$. Notice that $x_*(s = 0) = 1$.
Similarly to dimensional quantities one has expansions

$$\mathcal{Z} = \mathcal{Z}_0 + \sum_{k=1}^{s-1} s_k \mathcal{Z}_k + \frac{1}{2} \sum_{k_1,k_2=1}^{s-1} s_{k_1} s_{k_2} \mathcal{Z}_{k_1 k_2} + \dots$$
$$Y_u = Y_u^0 + \sum_{k=1}^{s-1} s_k Y_u^k + \frac{1}{2} \sum_{k_1,k_2=1}^{s-1} s_{k_1} s_{k_2} Y_u^{k_1 k_2} + \dots$$
$$Y_u^0(x) = C_0 x^{s+1} + C_0' x^{s-1} + \dots$$
$$Y_u^k(x) = C_k x^{s-k-1} + C_k' x^{s-k-3} + \dots$$
$$Y_u^{k_1 k_2}(x) = C_{k_1 k_2} x^{s-k_1-k_2-3} + C_{k_1 k_2}' x^{s-k_1-k_2-5} + \dots$$

Using and the partition function we find one- and two-point correlation numbers

$$\mathcal{Z}_k = \int_0^1 dx Y_u^0(x) Y_u^k(x)$$

$$\mathcal{Z}_{k_1 k_2} = \int_0^1 dx (Y_u^{k_1}(x) Y_u^{k_2}(x) + Y_u^0(x) Y_u^{k_1 k_2}(x))$$

where the one-point correlation numbers are singular only for even k and the two-point correlation numbers are singular only for even $k_1 + k_2$.
Fusion rules demand that one-point correlation numbers for $k \neq 0$ and two point numbers when $k_1 \neq k_2$ are zero.
The second term in the two point numbers is actually absent . Also it is convenient here to introduce a new variable y instead of x

$$\frac{y+1}{2} = x^2, \qquad dx = \frac{dy}{2\sqrt{2}(1+y)^{\frac{1}{2}}}$$

In terms of the variable y the polynomials
$Y_u^0, Y_u^k, \ldots$ will contain all powers instead of going with step 2 . And the shift was made in order for the interval of integration to be $[-1, 1]$ instead of $[0, 1]$.
Thus the fusion rules condition become an orthogonality condition on the polynomials Y_u^0, Y_u^k

$$\mathcal{Z}_k = \int_{-1}^1 \frac{dy}{2\sqrt{2}(1+y)^{\frac{1}{2}}} Y_u^0(x) Y_u^k(x) = 0, \quad k \neq 0$$

$$\mathcal{Z}_{k_1 k_2} = \int_{-1}^1 \frac{dy}{2\sqrt{2}(1+y)^{\frac{1}{2}}} Y_u^{k_1}(x) Y_u^{k_2}(x) = 0, \quad k_1 \neq k_2$$

Together with the condition $Y_u^0(1) = 0$, it determines the polynomials Y_u^0 and Y_u^k:

s odd	$Y_u^0(y) = P_{\frac{s+1}{2}}^{(0,-\frac{1}{2})}(y) - P_{\frac{s-1}{2}}^{(0,-\frac{1}{2})}(y)$
s even	$Y_u^0(y) = x\left(P_{\frac{s}{2}}^{(0,\frac{1}{2})}(y) - P_{\frac{s-2}{2}}^{(0,\frac{1}{2})}(y)\right)$
s+k odd	$Y_u^k(y) = P_{\frac{s-k-1}{2}}^{(0,-\frac{1}{2})}(y)$
s+k even	$Y_u^k(y) = xP_{\frac{s-k-2}{2}}^{(0,\frac{1}{2})}(y)$

where $P_n^{(a,b)}$ is Jacobi polynomial.

Due to the relation between Jacobi polynomials and Legendre polynomials P_n

$$P_n^{(0,-\frac{1}{2})}(2x^2 - 1) = P_{2n}(x)$$

$$xP_n^{(0,\frac{1}{2})}(2x^2 - 1) = P_{2n+1}(x)$$

it is of course in agreement with earlier results .

Taking the third derivatives of $\mathcal{Z}$ we get the three-point correlation numbers

$$\mathcal{Z}_{k_1k_2k_3} = -\frac{Y_u^{k_1}Y_u^{k_2}Y_u^{k_3}}{\frac{dY_u^0}{dx}}\Bigg|_{x=1} + \int_0^1 dx Y_u^{k_1k_2}Y_u^{k_3} = -\frac{1}{p} + \int_0^1 dx Y_u^{k_1k_2}Y_u^{k_3}$$

where we used the properties of Jacobi polynomials to get: $Y_u^k(x=1) = 1$, $\frac{dY_u^0}{dx}\big|_{x=1} = \frac{1}{2s+1} = \frac{1}{p}$. Besides we assume that $k_1, k_2 \leq k_3$.

As we will see the first term reproduces the expression from Minimal gravity and the role of the second term is to kill the first term when the fusion rules are violated.

Consider the case when $k_1 + k_2 + k_3$ is even. Fusion rules demand

$$\int_0^1 dx Y_u^{k_1k_2}Y_u^{k_3} = \begin{cases} \frac{1}{p} & \text{if} \quad k_1 + k_2 < k_3 \\ 0 & \text{if} \quad k_1 + k_2 \geq k_3 \end{cases}$$

Again, switching to variables y we get

$(s + k_1 + k_2)$ odd	$Y_u^{k_1k_2}(y) = \frac{1}{p}\sum_{n=0}^{\frac{s-k_1-k_2-3}{2}}(4n+1)P_n^{(0,-\frac{1}{2})}(y)$
$(s + k_1 + k_2)$ even	$Y_u^{k_1k_2}(y) = \frac{x}{p}\sum_{n=0}^{\frac{s-k_1-k_2-4}{2}}(4n+3)P_n^{(0,\frac{1}{2})}(y)$

Now we can compare the quantities independent on the normalization in the Douglas equation approach with those in MLG.

$$\frac{(\mathcal{Z}_{k_1 k_2 k_3})^2 \mathcal{Z}_0}{\prod_{i=1}^{3} \mathcal{Z}_{k_i k_i}}$$

When the fusion rules for three-point numbers are satisfied $Z_{k_1 k_2 k_3} = -\frac{1}{2s+1}$ and this quantity gives

$$\frac{\prod_{i=1}^{3}(2s - 2k_i - 1)}{(2s+3)(2s+1)(2s-1)}$$

which coincides with the value from Minimal Gravity .

Direct calculation of 4-point correlator gives

$$\mathcal{Z}_{k_1 k_2 k_3 k_4} = \left(-\frac{\frac{d^2 Y_u^0}{dx^2}}{(\frac{dY_u^0}{dx})^3} + \frac{\sum_{i=1}^{4} \frac{dY_u^{k_i}}{dx}}{(\frac{dY_u^0}{dx})^2} - \frac{\sum_{i<j} Y_u^{k_i k_j}}{\frac{dY_u^0}{dx}} \right) \Bigg|_{x=1} +$$
$$+ \int_0^1 dx (Y_u^{k_1 k_2} Y_u^{k_3 k_4} + Y_u^{k_1 k_3} Y_u^{k_2 k_4} + Y_u^{k_1 k_4} Y_u^{k_2 k_3}) +$$
$$+ \int_0^1 dx (Y_u^{k_1 k_2 k_3} Y_u^{k_4} + Y_u^{k_1 k_2 k_4} Y_u^{k_3} + Y_u^{k_1 k_3 k_4} Y_u^{k_2} + Y_u^{k_2 k_3 k_4} Y_u^{k_1})$$

The only role of the terms in the third line again is to satisfy the fusion rules by cancelling the unwanted terms in the first two lines when the fusion rules are violated.

$$\mathcal{Z}_{k_1k_2k_3k_4} = \left(-\frac{16\frac{d^2Y_u^0}{dy^2} + 4\frac{dY_u^0}{dy}}{(4\frac{dY_u^0}{dy})^3} + \frac{\sum_{i=1}^4 4\frac{dY_u^{k_i}}{dy}}{(4\frac{dY_u^0}{dy})^2} - \frac{\sum_{i<j} Y_u^{k_ik_j}}{4\frac{dY_u^0}{dy}} \right)\Bigg|_{y=1} +$$

$$+ \int_{-1}^{1} \frac{dy}{2\sqrt{2}(1+y)^{\frac{1}{2}}} (Y_u^{k_1k_2}Y_u^{k_3k_4} + Y_u^{k_1k_3}Y_u^{k_2k_4} + Y_u^{k_1k_4}Y_u^{k_2k_3})$$

Using the properties of Jacobi polynomials (for both even and odd s, k, k_1, k_2)

$$(Y_u^0)'(1) = \frac{2s+1}{4}, \qquad (Y_u^0)''(1) = \frac{(s-1)(s+2)(2s+1)}{32},$$

$$(Y_u^k)'(1) = \frac{(s-k-1)(s-k)}{8}, \qquad Y_u^{k_1k_2}(1) = \frac{(s-k_1-k_2-1)(s-k_1-k_2-2)}{2(2s+1)}$$

where prime denotes derivative with respect to y in agreement with MLG we obtain

$$\mathcal{Z}_{k_1k_2k_3k_4} = \frac{1}{2(2s+1)^2}\Big(-(s-1)(s+2) - 2 + \sum_{i=1}^{4} F(k_i+1) -$$

$$- F(k_{(12|34)}) - F(k_{(13|24)}) - F(k_{(14|23)})\Big)$$

where

$$F(k) = (s-k-1)(s-k-2), \qquad k_{(ij|lm)} = \min(k_i+k_j, k_l+k_m)$$

The generation function of MLG for $(2, 2s+1)$-series

$$\mathcal{Z} = \frac{1}{2}\int_0^{x_*} Y_u^2(x)dx$$

$$Y_u(x) = \sum_{N=0}^{\infty}\sum_{k_1\ldots k_N=1}^{s-1}\frac{s_{k_1}\ldots s_{k_N}}{N!}\left(\frac{d}{dx}\right)^{N-1}P_{s-\sum k-N}(x)$$

where $P_n(x)$ - Legendre polynomial , x_* - suitable solution of Srting equation

$$Y_u(x_*) = 0$$

The correlation numbers are given by the derivatives of the partition function

$$\langle O_{1,k_1+1}\ldots O_{1,k_N+1}\rangle = Z_{k_1\ldots k_n} = \frac{\partial}{\partial s_{k_1}}\ldots\frac{\partial}{\partial s_{k_n}}\Big|_{\substack{s_k=0\\ k\neq 0}}\mathcal{Z}$$

In the case $(3, p)$ where $p = 3s + p_0$,$p_0 = 1$ or 2 . Q and S are

$$Q = y^3 + uy + v, \qquad S(u,v) = Res\left(Q^{\frac{p}{3}+1} + \sum_{n=1}^{s}\tau_{1,n}Q^{s-n+\frac{p_0}{3}} + \sum_{n=s+1}^{p-1}\tau_{1,n}Q^{n-s-\frac{p_0}{3}}\right)$$

$$S(u,v) = S_{s+1,p_0} + \sum_{k=0}^{s-1} t_k S_{s-k-1,p_0} + \sum_{k=s}^{p-2} t_k S_{k-s+1,-p_0}$$

where by definition $t_k = \tau_{1,k+1}$ and $S_{k,\alpha} = ResQ^{k+\frac{\alpha}{3}}$.

One has to choose an appropriate solution (u_*, v_*) of the string equations

$$S_u = 0$$

$$S_v = 0$$

Using explicite expressions for $C^{\alpha}_{\beta\gamma}$ we find the free energy for this model

$$\mathcal{F} = \frac{1}{2}\int_{\Gamma(\lambda)}\left((S_u^2 - \frac{u}{3}S_v^2)du + 2S_uS_vdv\right)$$

where the contour $\Gamma(\lambda)$ goes from $(u,v) = (0,0)$ to $(u,v) = (u_*(\lambda), v_*(\lambda))$.

The scaling analysis gives

$$p \sim \mu^{\frac{1}{6}}, \quad u \sim \mu^{\frac{1}{3}}, \quad v \sim \mu^{\frac{1}{2}} \quad Z \sim \mu^{1+\frac{p}{3}}, \quad S \sim \mu^{\frac{p+4}{6}}$$

and the dimensions of the times t_k

$$t_k \sim \mu^{\frac{p+3-|p-3(k+1)|}{6}} \sim \begin{cases} \mu^{\frac{k+2}{2}}, & 0 \leq k \leq s-1 \\ \mu^{s-\frac{k}{2}+\frac{p_0}{3}}, & s \leq k \leq p-2 \end{cases}$$

The admissible resonances are

$$t_k = \lambda_k + c_k \mu^{\frac{k+2}{2}} + \sum_{\substack{l=1 \\ (k-l)\in 2Z}}^{s-1} \beta_{kl} \mu^{\frac{k-l}{2}} \lambda_l + \sum C_k^{k_1 k_2} \lambda_{k_1} \lambda_{k_2} + \ldots, \quad 0 \leq k \leq s-1$$

$$t_k = \lambda_k + \sum_{\substack{l=k+2 \\ (l-k)\in 2Z}}^{3s+\alpha-2} \beta_{kl} \mu^{\frac{l-k}{2}} \lambda_l + \sum C_k^{k_1 k_2} \lambda_{k_1} \lambda_{k_2} + \ldots, \quad s \leq k \leq p-2$$

Coefficients $C_k^{k_1 k_2}$ are non-zero only if $[\lambda_k] = [\lambda_{k_1}] + [\lambda_{k_2}]$
After substitution the times $t_k(\lambda)$ to free energy one gets the n-point correlation number as

$$Z_{k_1 \ldots k_n} = \frac{\partial}{\partial \lambda_{k_1}} \cdots \frac{\partial}{\partial \lambda_{k_n}} \Big|_{\lambda=0} \mathcal{F}$$

It will be crucial to know the solution (u^*, v^*) of the string equations at $\lambda_{m,n} = 0$. We will show that one of the equations is always satisfied by such a v^* that $v^*(\lambda = 0) = 0$.

$$S^0 \sim \mu^{\frac{p+4}{6}}$$

$$S^{k_1 \dots k_n} \sim \mu^{\frac{p+4}{6} - \sum_{i=1}^{n} \frac{p+3-|p-3(k_i+1)|}{6}}$$

Each of the functions $S^{k_1 \dots k_n}$ is a polynomial in the variables u, v, μ

$$S^{k_1 \dots k_n} = \sum_{M,N,K} u^M v^N \mu^K \sim \mu^{\frac{p+4}{6} - \sum_{i=1}^{n} \frac{p+3-|p-3(k_i+1)|}{6}}$$

$$\frac{M}{3} + \frac{N}{2} + K = \frac{p+4}{6} - \sum_{i=1}^{n} \frac{p+3-|p-3(k_i+1)|}{6}$$

It is equivalent to

$$p + \sum_{i=1}^{n} k_i - N = 2M + 2N + 6K + \sum_{i=1}^{n} (p + 3 - |p - 3(k_i + 1)| + k_i)$$

The right hand side is even. Thus $\left(p + \sum_{i=1}^{n} k_i - N\right)$ is also even. Consequently the functions $S^{k_1 \dots k_n}$ have definite parities with respect to v

$$S^0(u, -v) = (-1)^p S^0(u, v)$$

$$S^{k_1 \dots k_n}(u, -v) = (-1)^{p + \sum_i k_i} S^{k_1 \dots k_n}(u, v)$$

Thus $v^* = 0$ is always a solution at $\lambda = 0$

The contour of integration is following from $(0, 0)$ to $(u^*(\lambda = 0), v^*(\lambda = 0)) = (u_0, 0)$

The functions $S^{k_1\cdots k_n}(u,0)$ are polynomials in u and μ. Dimensional analysis gives

$$S^0(u,0) \sim u^{\frac{3(s+1)}{2}+\frac{1+p_0}{2}} + \dots$$

$$S^k(u,0) \sim \begin{cases} u^{\frac{3(s-k-1)}{2}+\frac{1+p_0}{2}} + \dots, & 1 \le k \le s-1 \\ u^{\frac{3(k-s+1)}{2}+\frac{1-p_0}{2}} + \dots, & s \le k \le p-2 \end{cases}$$

$$S^{k_1k_2}(u,0) \sim \begin{cases} u^{\frac{3(s-k_1-k_2-3)}{2}+\frac{1+p_0}{2}} + \dots, & 1 \le k_1,k_2 \le s-1 \\ u^{\frac{3(k_2-k_1-s-1)}{2}+\frac{1-p_0}{2}} + \dots, & 1 \le k_1 \le s-1, \quad s \le k_2 \le p-2 \end{cases}$$

$$S^{k_1k_2k_3}(u,0) \sim \begin{cases} u^{\frac{3(s-k_1-k_2-k_3-5)}{2}+\frac{1+p_0}{2}} + \dots, & 1 \le k_1,k_2,k_3 \le s-1 \\ u^{\frac{3(k_3-k_1-k_2-s-3)}{2}+\frac{1-p_0}{2}} + \dots, & 1 \le k_1,k_2 \le s-1, \quad s \le k_3 \le p-2 \\ u^{\frac{3(k_2+k_3-k_1-3s-1)}{2}+\frac{1-3p_0}{2}} + \dots, & 1 \le k_1 \le s-1, \quad s \le k_2,k_3 \le p-2 \\ u^{\frac{3(k_1+k_2+k_3-5s+1)}{2}+\frac{1-5p_0}{2}} + \dots, & s \le k_1,k_2,k_3 \le p-2 \end{cases}$$

When evaluating one- and two-point numbers, we do not need to differentiate over the limits of integration since these terms give zero because of the string equation.

	p even	p - odd
	$Z_0 = \frac{1}{2}\int_0^{u_0}(S_u^0)^2 du$	$Z_0 = -\frac{1}{6}\int_0^{u_0}(S_v^0)^2 udu$
	$Z_k = \int_0^{u_0} S_u^0 S_u^k du$	$Z_k = -\frac{1}{3}\int_0^{u_0} S_v^0 S_v^k udu$
k_1, k_2 even	$Z_{k_1 k_2} = \int_0^{u_0}(S_u^{k_1} S_u^{k_2} + S_u^0 S_u^{k_1 k_2})du$	$Z_{k_1 k_2} = -\frac{1}{3}\int_0^{u_0}(S_v^{k_1} S_v^{k_2} + S_v^0 S_v^{k_1 k_2})udu$
k_1, k_2 odd	$Z_{k_1 k_2} = \int_0^{u_0}(S_u^0 S_u^{k_1 k_2} - \frac{u}{3} S_v^{k_1} S_v^{k_2})du$	$Z_{k_1 k_2} = \int_0^{u_0}(S_u^{k_1} S_u^{k_2} - \frac{u}{3} S_v^0 S_v^{k_1 k_2})du$
$k_1 + k_2$ odd	$Z_{k_1 k_2} = 0$	$Z_{k_1 k_2} = 0$

where all the polynomials are taken at $v = 0$ since the contour of integration goes along the u-axis.

Now the analysis is similar to the case of Lee-Yang series $(2, 2s+1)$. We switch from $S(u, 0)$ and u to dimensionless quantities $Y(x)$ and $x = \frac{u}{u_0}$ respectively

$$\frac{y+1}{2} = x^3, \qquad dx = \frac{dy}{3\sqrt[3]{2}(1+y)^{\frac{2}{3}}}$$

Then from the selection rules we get

p even	$Y_u^0 = x^{2(p_0-1)}(P^{(0,\frac{2}{3}(2p_0-3))}_{\frac{s-p_0+2}{2}}(y) - P^{(0,\frac{2}{3}(2p_0-3))}_{\frac{s-p_0}{2}}(y))$
p odd	$Y_v^0 = x^{2-p_0}(P^{(0,\frac{2}{3}(1-p_0))}_{\frac{s+p_0-1}{2}}(y) - P^{(0,\frac{2}{3}(1-p_0))}_{\frac{s+p_0-3}{2}}(y))$
$(p+k)$ even, $k < s$	$Y_u^k = x^{2(p_0-1)}P^{(0,\frac{2}{3}(2p_0-3))}_{\frac{s-k-p_0}{2}}(y)$
$(p+k)$ odd, $k < s$	$Y_v^k = x^{2-p_0}P^{(0,\frac{2}{3}(1-p_0))}_{\frac{s-k+p_0-3}{2}}(y)$
$(p+k)$ even, $k \geq s$	$Y_u^k = x^{2(2-p_0)}P^{(0,\frac{2}{3}(3-2p_0))}_{\frac{k-s+p_0-2}{2}}(y)$
$(p+k)$ odd, $k \geq s$	$Y_v^k = x^{p_0-1}P^{(0,\frac{2}{3}(p_0-2))}_{\frac{k-s-p_0+1}{2}}(y)$

where all the polynomials are evaluated at $v = 0$.

For the three-point correlation numbers a direct calculation gives

$$Z_{k_1k_2k_3} = (S_u^{k_1}S_u^{k_2} - \frac{u_0}{3}S_v^{k_1}S_v^{k_2})\partial_{k_3}u_* + S_u^{k_1}S_v^{k_2}\partial_{k_3}v_* +$$
$$+ \int_0^{u_0} du(S_u^{k_1}S_u^{k_2k_3} - \frac{u}{3}S_v^{k_1}S_v^{k_2k_3}) + \int_0^{u_0} du(S_u^0 S_u^{k_1k_2k_3} - \frac{u}{3}S_v^0 S_v^{k_1k_2k_3}) + perm.$$

Taking derivative of the string equations one gets at $s_k = 0$

$$S_u^k + S_{uu}^0\partial_k u_* + S_{uv}^0\partial_k v_* = 0$$

$$S_v^k + S_{vu}^0\partial_k u_* + S_{vv}^0\partial_k v_* = 0$$

So using the parities of polynomials $S^{k_1\cdots k_n}$ one has at $v = 0$

	p even	p odd
k even	$\partial_k u_* = -\frac{S_u^k}{S_{uu}^0}$	$\partial_k u_* = -\frac{S_v^k}{S_{uv}^0}$
k odd	$\partial_k v_* = -\frac{S_v^k}{S_{vv}^0}$	$\partial_k v_* = -\frac{S_u^k}{S_{uv}^0}$

where for each case only non-zero derivative is represented.
For instance for even p and k we have $\partial_k v_* = 0$.

The simplest 3-point case to analyse here is when p is even and all k are even

$$Z_{k_1k_2k_3}=-\frac{S_u^{k_1}S_u^{k_2}S_u^{k_3}}{S_{uu}^0}\Big|_{u=u_0}+\int_0^{u_0}S_u^0S_u^{k_1k_2k_3}du+\int_0^{u_0}(S_u^{k_1}S_u^{k_2k_3}+S_u^{k_2}S_u^{k_1k_3}+S_u^{k_3}S_u^{k_1k_2})du.$$

- $1\leq k_1,k_2,k_3\leq s-1$; p,k – even. In this 3-point correlators are singular .

The expression for them reduces to (we assume that $k_3>k_1,k_2$)

$$\mathcal{Z}_{k_1k_2k_3}=-\frac{Y_u^{k_1}Y_u^{k_2}Y_u^{k_3}}{(Y_u^0)'}\Big|_{x=1}+\int_0^1Y_u^{k_3}Y_u^{k_1k_2}dx$$

Two other integral terms are absent

Thus to satisfy fusion rules one needs the following condition to hold

$$\int_0^1Y_u^{k_3}Y_u^{k_1k_2}dx=\begin{cases}0, & \text{if}\quad k_3\leq k_1+k_2\\ \frac{1}{p}, & \text{if}\quad k_3>k_1+k_2\end{cases}$$

This determines

$$Y_u^{k_1k_2}=\frac{1}{p}\sum_{k=0}^{\frac{s-k_1-k_2-p_0-2}{2}}(6k+4p_0-3)x^{2(p_0-1)}P_k^{(0,\frac{2}{3}(2p_0-3))},\qquad 1\leq k_1,k_2\leq s-1$$

We arrive at the expression for nonzero three-point correlation numbers

$$\mathcal{Z}_{k_1k_2k_3}=-\frac{Y_u^{k_1}Y_u^{k_2}Y_u^{k_3}}{(Y_u^0)'}\Big|_{x=1}=-\frac{1}{p}$$

We can evaluate non-vanishing correlation numbers, when selection rules are satisfied .

In this case the three-point correlation numbers are given by the first term in the formula above

$$Z_0 = \frac{p}{(p+3)(p-3)}$$

$$Z_{kk} = \begin{cases} \frac{1}{p-3(k+1)}, & 1 \leq k \leq s-1, \quad k \text{ even} \\ \frac{1}{3(k+1)-p}, & s \leq k \leq p-2, \quad k \text{ even} \end{cases}$$

$$Z_{k_1 k_2 k_3} = -\frac{1}{p}, \qquad 1 \leq k_i \leq p-2, \quad k \text{ even}.$$

The quantity that doesn't depend on the normalization of the operators and correlators is

$$\frac{(Z_{k_1 k_2 k_3})^2 Z_0}{\prod_{i=1}^{3} Z_{k_i k_i}} = \frac{\prod_{i=1}^{3} |p - k_i q|}{p(p+q)(p-q)}$$

where $p = 3s + p_0, q = 3$.

It is in agreement with the direct calculation of the integrals over the moduli space in Minimal Gravity .

Conclusion

We have argued that Douglas string equation approach being accompanied by the suitable resonance transformation of the coordinates is applicable to Liouville Minimal Gravity.

Useful representation for the partition function of the Liouville Minimal Gravity perturbed by primary operators is suggested .

We found an appropriate solution of the Douglas string equation in the $(2, 2s+1)$ and case of $(3, 3s+p_0)$ cases and the resonance relations which lead to the correlation numbers obeying MLG fusion rules.

Using this input we managed to make full correspondence with $(3.3s+p_0)$ Liouville Minimal Gravity up to two-point correlation numbers.

The three- and four-point correlation numbers derived from the Douglas string equation also coincide with the direct calculations in $(q,p) = (3, 3s+p_0)$ MLG when the selection rules are satisfied .

In the case when the three- and four-point correlation numbers have to vanish we obtained only partial agreement with Liouville Minimal Gravity.
It is an open problem ...