Topological string theories versus matrix models

R. Flume, Bonn FRG

There are various representations of topological string theories (TST)s through matrix models (MM)s. Among those I concentrate on the ones leading to the conceivably simplest MMs: hermitean 1-matrix models.

a) TSTs

It is well known through the work of Bershadsky, Cecotti, Ooguri and Vafa that the free energy of TSTs given in terms of a genus expansion

$$F = \sum_{g \geq 0} (g_s)^{2g-2} F^{(g)}$$

displays holomorphic and antiholomorphic dependence on the string moduli $F^g = F^g(t,\bar{t})$, t and $\bar{t}$ denoting collectively the moduli of the TST.

$F^g(t,\bar{t})$ is invariant under some special group of modular transformations which guarantees that the choice of a homology basis — to be made in intermediate steps of the evaluation of $F^{(g)}$ — is irrelevant.

A holomorphic amplitude can be isolated in the limit

$$F^{(g)}(t) = \lim_{\bar{t}\to\infty} F^g(t,\bar{t})$$

but $F^{(g)}$ is not modular invariant.

Bershadsky et al. found a system of equations, so called anomaly equattions describing the anholomorphic $\bar{t}$ - dependendence:

$$\partial^{\bar t} F^{(g)} = C_{\bar t}^{tt}\left(D_t D_t F^{(g-1)} + \sum_{r=1}^{g-1} D^t F^{(r)} D^t F^{(g-r)}\right)$$

$C_{\bar t}^{tt}$ denotes here some tensor and D_t stands for a certain covariant derivative.

That equations can easily be resolved recursively if a finite, concretely known basis of modular forms is available. The remaining holomorphic dependence of $F^{(g)}$ – sometimes called holomorphic ambiguity – can be fixed in favourable situations by imposing suitable boundary conditions. But this elegant procedure only works in a very limited number of situations.

b.) MMs

The partition function and free energy ~~and~~ of the 1-MM of Hermitian matrices is

$$Z_N = \int \prod_{i,j} dM_{ij} \cdot e^{-\mathrm{tr}\, V(\varphi)} ,$$

$$V(\varphi) = \sum_k g_k \varphi^k ,$$

$$Z_N = \exp N^2 F(N, g) .$$

The $1/N$ expansion of $F(N,g)$ is supposed to match the genus expansion of a certain TST's free energy. The basic tool to achieve the $1/N$ expansion are the so called loop equations.

It is convenient to determine first the 1-point correlator of the resolvent operator and proceed then through a straightforward integration to the free energy. In leading order of $1/N$ the loop equations give rise to a quadratic equation with the solution

$$W_1^{(0)}(p) = \frac{1}{2} V'(p) - \frac{1}{2}\sqrt{V'^2(p) + P(p)}$$

where $W_1^{(0)}$ denotes ~~denotes~~ the 1-point correlator in question to leading order in $1/N$ and P being a polynomial to be determined by boundary conditions. To the formula for $W_1^{(0)}$ is attached the hyperelliptic equation

$$y^2(p) = V'^2(p) + P(p)$$

with the solution

$$y(p) = M(p) \prod_{i=1}^{2s} (p - x_i)$$

where $M(p)$ denotes a a polynomial and we suppose that the hyperelliptic geometry displays $2s$ branch points $x_1, \ldots, x_{2s}$.

The ingredients to find with the loop equations the higher order contributions to $W_1(p)$ and then also to the free energy are first of all the above hyperelliptic function y – in proper mathematical terms a 1-form – and second a 2-form $B(x, y)$ on the hyperelliptic Riemann surface – often called a Bergman kernel ~~-~~ with the defining properties

$B(x,y) = B(y,x)$, $B(x,y) \sim \frac{1}{(x-y)^2}$ for $x \to y$.

It can be shown that the higher order contributions to the resolvent correlator and the free energy can be encoded into the Feynman graphs of an effective field theory concentrated on the above branch points $x_1 \ldots x_{2s}$. The role of propagators play the Bergmann kernel. The vertices of the graphs are decorated by $1/y$ and its derivatives.

c) TST versus HM

One of the obstacles to make use of HMs for TST is that HMs ostentatiously lack modular invariance, important for TST. From the 2s branch points $x_1 \ldots x_{2s}$ one has to select s intervals, e.g. $[x_1, x_2], \ldots [x_{2s-1}, x_{2s}]$ as supports of eigenvalues. So one is making a definite choice within the homology of the hyper-elliptic Riemann surface which is in conflict with a presumed modular invariance. A cure for this deficiency was devised by Eynard and Orantin and further developed by these authors with Marinko.

They propose to consider a modified HH by substituting

$$B(x,y) \to B(x,y) + \sum_{ij} da_i(x)\, \mathcal{X}_{ij}\, da_j(y)$$

with $\{da_i\}$ denoting a basis of holomorphic 1-forms and $\mathcal{X}$ being a symmetric matrise.

It can be shown that modular transformations are absorbed into a modification of $\mathcal{X}$.

Making in particular the specific choice $\mathcal{X} \simeq \mathrm{Im}(\frac{1}{\tau})$

with τ denoting the modular matrix of the hyperelliptic surface in question the HH becomes form invariant ~~under~~ under modular transformations. With that modification of the HH at hand one can start to compare standard HH calculations with TST inspired attempts via modular forms. I have been able to perform the program for the simplest conceivable non-trivial model, a model with ~~a~~ cubic potential and a symmetric choice of the then four branch points in question. The most pedestrian way to proceed consists in expressing the function y and the Bergmann kernel in terms of modular forms and

insert into the expressions for the mentioned Feynman graphs. In this way a complete equivalence of the two approaches for the special case of a cubic potential is established.
That result should of course give rise to some obvious generalisations. The first consists in considering a QM with an arbitrary polynomial potential which will give rise to an arbitrary number of brand points, that is a general hyperelliptic geometry – in contrast to the up to now considered elliptic geometry.