M5-brane superconformal indices

Seok Kim
(Seoul National University)

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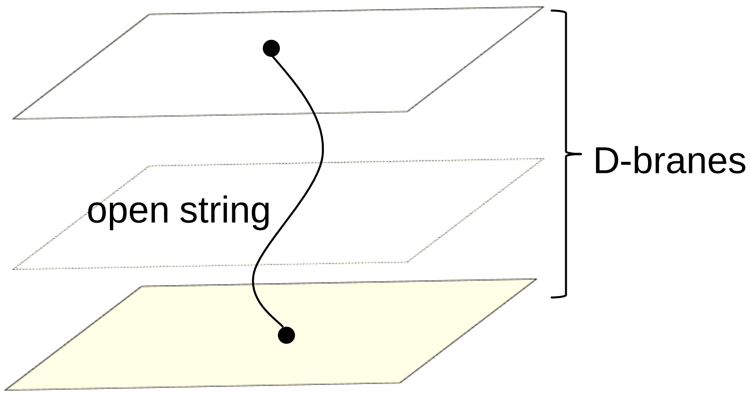
talk based on:
Hee-Cheol Kim, S.K., Sung-Soo Kim, Kimyeong Lee,
"The general M5-brane superconformal index," arXiv:1307.7660

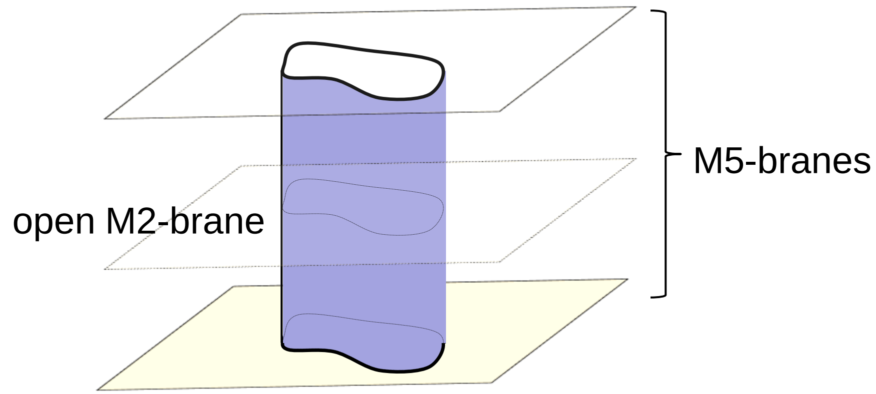
The following works are closely related, but not mentioned in this talk:
Hee-Cheol Kim, S.K. arXiv:1206.6339; Lockhart, Vafa, arXiv:1210.5909;
Hee-Cheol Kim, Joonho Kim, S.K. arXiv:1211.0144.

M5-branes and 6d (2,0) SCFT

- M5-branes host 6d (2,0) SCFT: very unique quantum field theory.
- Not like the familiar Yang-Mills. It should be a “tensor gauge theory”

$$\int_{\Sigma_1} A_\mu dx^\mu \qquad\qquad \int_{\Sigma_2} B_{\mu\nu} dx^\mu dx^\nu$$





- N^3 light degrees of freedom should live on N M5’s.

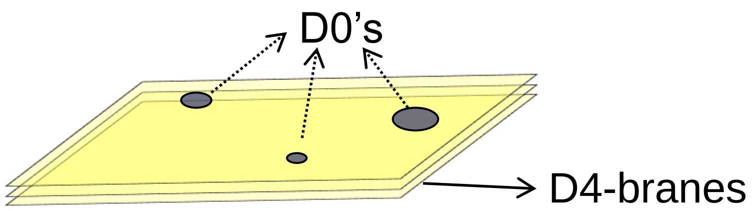
- Its compactification leads to interesting quantum systems.
- “M-theory of QFT.” A unifying framework to understand QFT dualities

Circle compactification and 5d SYM

- However, we know almost nothing about its microscopic definition.

- Compactify on S^1, 5d (S)YM at low E: non-perturbative study, some 6d physics.
- Instanton solitons remember the 6d physics: D0's on D4's = KK modes

$F_{\mu\nu} = \star_4 F_{\mu\nu}$ on R^4 $\qquad \frac{4\pi^2}{g_{YM}^2} = \frac{1}{r_1}$



- BPS quantities are often calculable in non-renormalizable low E theories.
- This talk: 6d (2,0) theory on $S^5 \times S^1$ from SYM on S^5 or $CP^2 \times S^1$.

I will mostly focus on this approach today.

※ (2,0) theory on other manifolds: $R^4 \times T^2$ [H.Kim, S.K, E.Koh, K.Lee, S.Lee] [Haghighat, Iqbal, Kozcaz, Lockart, Vafa], $S^3 \times S^1 \times M_2$ [Fukuda, Kawano, Matsumiya], $S^2 \times S^1 \times M_3$ [Yagi] [Lee, Yamazaki], $S^3 \times M_3$ [Cordova, Jafferis]; probably more to come

QFT on CP^2 x R

- (2,0) theory on S^5 x R: impose an extra Z_K orbifold, fractional shift on Hopf fiber
- SUSY KK reduction on S^1/Z_K fiber: energy E ; SO(6) j_1, j_2, j_3 ; $SO(5)_R$ R_1, R_2

2π/K rotation with $$k \equiv j_1 + j_2 + j_3 + \frac{3}{2}(R_1 + R_2) + n(R_1 - R_2)$$

- Half-an-odd integer n: twisted reductions, infinitely many 5d QFT
- Our interest: strong-coupling QFT at K=1: instantons provide KK towers
- On-shell (Euclidean) action: constrained by Abelian 5d reduction

$$S = \frac{1}{\tilde{g}_{YM}^2}\int d^5x\sqrt{g}\,\mathrm{tr}\Big[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi^I D^\mu\phi^I - \frac{i}{2}\lambda^\dagger\gamma^\mu D_\mu\lambda - \frac{1}{4}[\phi^I,\phi^J]^2 - \frac{i}{2}\lambda^\dagger\hat{\gamma}^I[\lambda,\phi^I]$$
$$+\frac{2}{r^2}(\phi_I)^2 - \frac{1}{2r^2}(M_n\phi^I)^2 + \frac{1}{8r}\lambda^\dagger J_{\mu\nu}\gamma^{\mu\nu}\lambda - \frac{i}{2r}\lambda^\dagger M_n\lambda - \frac{i}{r}(3-2n)[\phi^1,\phi^2]\phi^3 - \frac{i}{r}(3+2n)[\phi^4,\phi^5]\phi^3$$
$$-\frac{i}{2r\sqrt{g}}\epsilon^{\mu\nu\lambda\rho\sigma}\left(A_\mu\partial_\nu A_\lambda - \frac{2i}{3}A_\mu A_\nu A_\lambda\right)J_{\rho\sigma}\Big]$$

R_1 R_2

$$\tilde{g}_{YM}^2 = 4\pi^2 r/K \qquad M_n \equiv \tfrac{3}{2}(R_1+R_2) + n(R_1 - R_2)$$

- Symmetry: SU(3|1) x SU(1|1) at n = ½, - ½ (8 manifest SUSY); SU(1|2) at n = 3/2, -3/2 (4 manifest); SU(1|1) (2 manifest) otherwise. All of them are subgroups of OSp(8*|4).

Off-shell action & SUSY

- action with "off-shell SUSY": 5d N=2 vector: N=1 vector + adjoint hyper

$$\tilde{g}_{YM}^2\mathcal{L}_V = \mathrm{tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi D^\mu\phi - \frac{i}{2}\chi^\dagger\gamma^\mu D_\mu\chi - \frac{i}{2}\chi^\dagger[\phi,\chi] - \frac{1}{2}\left(D^I + \frac{3}{r}\phi\delta_3^I\right)^2 \right.$$
$$\left. + \frac{2}{r^2}\phi^2 + \frac{1}{8r}\chi^\dagger J_{\mu\nu}\gamma^{\mu\nu}\chi - \frac{i}{2r}\chi^\dagger M_n\chi - \frac{i}{2r\sqrt{g}}\epsilon^{\mu\nu\chi\rho\sigma}\left(A_\mu\partial_\nu A_\chi - \frac{2i}{3}A_\mu A_\nu A_\chi\right)J_{\rho\sigma}\right]$$

$$\tilde{g}_{YM}^2\mathcal{L}_H = \mathrm{tr}\left[|D_\mu q^A|^2 + \frac{4}{r^2}|q^A|^2 + \frac{1}{r^2}|M_n q^A|^2 + \left|[\phi,q^A]\right|^2 + D^I(\sigma^I)^A{}_B[q^B,\bar{q}_A] - \frac{2n}{r}\phi[q^A,\bar{q}_A] - \bar{F}_{A'}F^{A'} \right.$$
$$\left. -i\psi^\dagger\gamma^\mu D_\mu\psi + i\psi^\dagger[\phi,\psi] + \sqrt{2}i\psi^\dagger[\chi_A,q^A] - \sqrt{2}i[\bar{q}_A,\chi^{\dagger A}]\psi + \frac{1}{4r}\psi^\dagger J_{\mu\nu}\gamma^{\mu\nu}\psi - \frac{i}{r}\psi^\dagger M_n\psi\right]$$

- SUSY:

$$\delta\chi = \frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon + iD_\mu\phi\gamma^\mu\epsilon - iD^I\sigma^I\epsilon - \frac{i}{r}\phi\sigma^3\epsilon$$
$$\delta\phi = -\epsilon^\dagger\chi\ , \qquad \delta A_\mu = i\epsilon^\dagger\gamma_\mu\chi$$
$$\delta D^I = \epsilon^\dagger\sigma^I\gamma^\mu D_\mu\chi - \frac{i}{4r}\epsilon^\dagger\gamma^{\mu\nu}J_{\mu\nu}\sigma^I\chi + \epsilon^\dagger\sigma^I[\phi,\chi] - \epsilon^\dagger\frac{1}{2r}\sigma^3\sigma^I\chi$$

$$\delta q^A = \sqrt{2}\epsilon^{\dagger A}\psi$$
$$\delta\psi = \sqrt{2}\left[-iD_\mu q_A\gamma^\mu\epsilon^A - i[\phi,q_A]\epsilon^A + \frac{i}{r}(M_n q_A)\epsilon^A - \frac{2i}{r}q_A\eta^A - i\hat{\epsilon}^{A'}F_{A'}\right]$$
$$\delta F^{A'} = \sqrt{2}(\hat{\epsilon}^\dagger)^{A'}\left[\gamma^\mu D_\mu\psi - [\phi,\psi] - \sqrt{2}[\chi_A,q^A] + \frac{i}{4r}J_{\mu\nu}\gamma^{\mu\nu}\psi + \frac{1}{r}M_n\psi\right]$$

Indices for 6d (2,0) theories

- (2,0) theory on $S^5 \times R$: energy E ; $SO(6)$ j_1, j_2, j_3 ; $SO(5)_R$ R_1, R_2

- Choose a pair of Q, S (= Q^+): in SU(1|1) of the 5d QFT

$$Q^{(R_1,R_2)}_{(j_1,j_2,j_3)} \to Q^{(\frac{1}{2},\frac{1}{2})}_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} \ : \ \text{BPS bound } E = 2R_1 + 2R_2 + j_1 + j_2 + j_3$$

[Kinney,Maldacena,Minwalla,Raju] [Bhattacharya,Bhattacharyya,Minwalla,Raju]

- Index partition function on $S^5 \times S^1$: counts local BPS operators on R^6

$$I(\beta, m, a_i) = \text{Tr}\left[(-1)^F e^{-\beta'\{Q,S\}} e^{-\beta(E-\frac{R_1+R_2}{2})} e^{\beta m(R_1-R_2)} e^{-\beta(aj_1+bj_2+cj_3)}\right]$$

- Comes with 4 chemical potentials. $(a_i) = (a,b,c)$ satisfying $a+b+c=0$: $U(1)^2$

- In 5d, one computes the path integral on $CP^2 \times S^1$ with twisted boundary conditions, preserving at least 2 SUSY.

- Instantons contribute to the BPS energy: "central charge" of superalgebra, within the SUSY algebra classically realized in 5d (but NOT central in OSp(8*|4)…)

Index on $CP^2 \times S^1$

- Localization:

$$Z(\beta) = \int e^{-S-tQV} \;:\; t \text{ independent}$$

V chosen to satisfy $[Q^2, V] = 0$

- Q-exact deformation: $V = (\delta\chi)^\dagger \chi + \delta((\delta\psi)^\dagger \psi + \psi^\dagger(\delta\psi^\dagger)^\dagger)$

$$(\delta\chi)^\dagger \equiv -\frac{1}{2}\epsilon^\dagger\gamma^{\mu\nu}F_{\mu\nu} - i\epsilon^\dagger\gamma^\mu D_\mu\phi - \epsilon^\dagger\sigma^a(iD^a) + i\epsilon^\dagger\sigma^3\left(\frac{1-\xi}{r}\phi - (D + \frac{\xi}{r}\phi)\right)$$

- bosonic part (vector):

parameter labeling a complex deformation of ϕ path integral contour: fine with any $\xi > 1$

$$\left(F^-_{mn} - \frac{1-\xi}{2r}\phi J_{mn}\right)^2 - (D^a)^2 - \left(D + \frac{\xi}{r}\phi\right)^2 + (D_\mu\phi)^2 + (F_{m\tau})^2$$

- Saddle point conditions: and all hyper fields & fermions are 0

$$D^1 = D^2 = 0\;,\quad F^- = \frac{2s}{r^2}J\;,\quad \frac{\phi}{r} + D = \frac{4s}{r^2}\;,\quad D + \frac{\xi}{r}\phi = 0$$

anti-self-dual instantons allowed on CP^2, proportional to Kahler 2-form

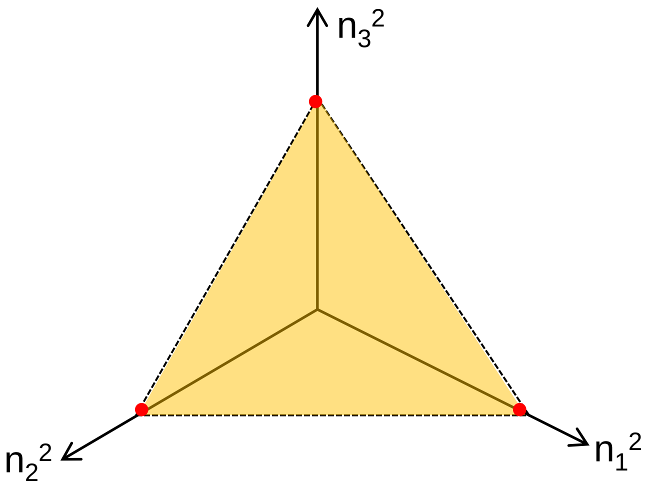
- holonomy on S^1, in the same Cartan as scalar and anti-self-dual flux “s”

$$\oint_{S^1} A = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)\;,\quad \lambda_i \sim \lambda_i + 2\pi$$

Index on $CP^2 \times S^1$

- Self-dual instantons: generally, come with a moduli space.
- In our problem, all of them lifted after turning on chemical potentials:

- $U(1)^2$ on CP^2: self-dual instanton profiles are singularly localized at 3 fixed points of CP^2. So position & "scale" moduli lifted.
- Near 3 fixed points, our QFT is exactly Nekrasov's Ω-deformed QFT on $R^4 \times S^1$



$$F_{m\tau} = \frac{i}{2}(1-\xi)\phi(a_i n_i dn_i \wedge d\tau)_{m\tau}$$

$$\mathbb{C}^3 : \ Z_i = n_i e^{i\phi_i} \ , \ n_1^2 + n_2^2 + n_3^2 = 1 \ , \ (i = 1, 2, 3)$$

$$(n_1, n_2, n_3) = (1,0,0), \ (0,1,0) \ \text{or} \ (0,0,1)$$

- Holonomy & anti-self-dual flux: gauge orientation frozen to Cartans $U(1)^N$ in $U(N)$.

(F^+ and F^- are localized/delocalized in CP^2, respectively: we "reasonably" assume that self-dual instantons can be "superposed" with the anti-self-dual fluxes in the same $U(1)^N$.)

Result

- Determinant: factorized to contributions from 3 fixed points on CP^2.
- Result: $$\frac{1}{N!}\sum_{s_1,s_2,\cdots s_N=-\infty}^{\infty}\oint\left[\frac{d\lambda_i}{2\pi}\right]e^{\frac{\beta}{2}\sum_{i=1}^N s_i^2 - i\sum_i s_i\lambda_i} Z^{(1)}_{\rm pert}Z^{(1)}_{\rm inst}\cdot Z^{(2)}_{\rm pert}Z^{(2)}_{\rm inst}\cdot Z^{(3)}_{\rm pert}Z^{(3)}_{\rm inst}$$
- Classical action: $$S_0 = -\frac{\beta}{2}\sum_i s_i^2 + i\sum_i s_i(\lambda_i + i\sigma_i)$$ → from $\sim \int J\wedge(AdA - \frac{2i}{3}A^3)$

  coincides with the instanton number from the anti-self-dual flux: $$\frac{1}{8\pi^2}\int_{CP^2}{\rm tr}(F\wedge F) = -\frac{1}{2}\sum_i s_i^2$$
- $Z_{\rm pert}$: $$Z^{(1)}_{\rm pert}Z^{(2)}_{\rm pert}Z^{(3)}_{\rm pert} = \prod_{\alpha\in\Delta_+}\frac{\prod_{\sum_{i=1}^3 p_i=\alpha(s)} 2\sin\frac{\alpha(\lambda+i\sigma)+\beta p_i a_i}{2}\cdot\prod_{\sum_{i=1}^3 p_i=\alpha(s)-3} 2\sin\frac{\alpha(\lambda+i\sigma)+\beta p_i a_i}{2}}{\prod_{\sum_{i=1}^3 p_i=\alpha(s)-1} 2\sin\frac{\alpha(\lambda+i\sigma)+\beta p_i a_i-\beta\hat{m}}{2}\cdot\prod_{\sum_{i=1}^3 p_i=\alpha(s)-2} 2\sin\frac{\alpha(\lambda+i\sigma)+\beta p_i a_i+\beta\hat{m}}{2}}$$
- $Z_{\rm inst}$: product of 3 Nekrasov's $Z_{\rm inst}$ on $R^4 \times S^1$, with suitable identifications of parameters

$$(n_1,n_2,n_3)=(1,0,0)\ :\ \epsilon_1=b-a,\ \epsilon_2=c-a,\ m_0=m+n(1+a)$$
$$(n_1,n_2,n_3)=(0,1,0)\ :\ \epsilon_1=c-b,\ \epsilon_2=a-b,\ m_0=m+n(1+b)$$
$$(n_1,n_2,n_3)=(0,0,1)\ :\ \epsilon_1=a-c,\ \epsilon_2=b-c,\ m_0=m+n(1+c)$$

→ mass of 5d N=1* theory on $R^4 \times S^1$

$$S_0=\beta k_i(1+a_i)\ :\ k_i \text{ self dual instantons at fixed points}$$

(There is an integration contour issue for λ variables, on going around poles. A suitable "i ε" prescription from the superconformal unitarity bounds determines it.)

A test: Abelian index

- As the index is known from free 6d tensor theory, this exercise provides a check of our 5d result. Also, it explains a subtle nature of our study: “small instantons”

- The U(1) partition function on $R^4 \times S^1$ [Iqbal, Kozcaz, Shabbir] [Awata, Kanno]:

$$Z_{\rm pert} = PE\left[\frac{1}{2}I_+(\epsilon_{1,2}, m_0)\right] \ , \quad Z_{\rm inst} = PE\left[I_-(\epsilon_{1,2}, m_0)\frac{e^{-4\pi^2 r_1/g_{YM}^2}}{1-e^{-4\pi^2 r_1/g_{YM}^2}} \right]$$

$$I_\pm \equiv \frac{\sinh\frac{\beta(m_0+\epsilon_\pm)}{2}\sinh\frac{\beta(m_0-\epsilon_\pm)}{2}}{\sinh\frac{\beta\epsilon_1}{2}\sinh\frac{\beta\epsilon_2}{2}} \qquad PE[f(x)] \equiv \exp\left[\sum_{n=1}^\infty \frac{1}{n}f(nx)\right]$$

- Trivial contour integral & anti-self-dual flux sum: just multiply 3 factors.

- Combine 3 such contributions: the exponents add to be

$$\left[\frac{1}{2}I_+ + I_-\frac{e^{-\beta(1+a)}}{1-e^{-\beta(1+a)}}\right]_{\rm 1st} + \left[\cdots\right]_{\rm 2nd} + \left[\cdots\right]_{\rm 3rd} + \boxed{1}$$

contribution from a bosonic zero mode in the FP ghost multiplet [Pestun]

$$\longrightarrow \frac{e^{-\frac{3}{2}\beta}(e^{\beta m}+e^{-\beta m}) - e^{-2\beta}(e^{\beta a}+e^{\beta b}+e^{\beta c}) + e^{-3\beta}}{(1-e^{-\beta(1+a)})(1-e^{-\beta(1+b)})(1-e^{-\beta(1+c)})}$$

PE[ ] of this is the correct 6d Abelian index
[Bhattacharya, Bhattacharyya, Minwalla, Raju]

Non-Abelian index

- To use semi-classical instanton expansion of Nekrasov, set 4 fugacities to obey:

keep O(1)

$$q = e^{-\beta} \ll 1 \,, \quad \underbrace{y = e^{\beta \hat{m}} = e^{\beta(m+n)} \,, \quad y_i = e^{-\beta a_i}}$$

- q is conjugate to instanton charge ~ "energy level" of 6d states:

$$\begin{aligned} k &\equiv \frac{1}{8\pi^2}\int_{\mathbb{CP}^2} \mathrm{tr} F\wedge F = \frac{1}{8\pi^2}\int_{\mathbb{CP}^2} \mathrm{tr} F^+\wedge F^+ + \frac{1}{8\pi^2}\int_{\mathbb{CP}^2} \mathrm{tr} F^-\wedge F^- \\ &= k_{SD} + \frac{1}{2\pi^2}\sum_{i=1}^N s_i^2 \int_{\mathbb{CP}^2} J\wedge J = k_{SD} - \frac{1}{\pi^2}\sum_{i=1}^N s_i^2\, \mathrm{vol}(\mathbb{CP}^2) = k_{SD} \boxed{-\frac{1}{2}\sum_{i=1}^N s_i^2} \end{aligned}$$

- Vacuum: k_{SD} =0 and k_{ASD} = - $N(N^2-1)/6$ from

$$s = (s_1, s_2, \cdots, s_N) = (N-1, N-3, N-5, \cdots, -(N-1))$$

- Excitations: either increase k_{SD} or "decrease" the s flux (towards s=0)

convention from here: "k=0" for vacuum, k>0 for excitations (subtract the vacuum value of k)

Finite N & large N indices

(from here, we only study the 5d QFT with n = -½ , with 8 manifest SUSY)

- The vacuum (k=0):

$$I_{k=0} = e^{\beta(1-\hat{m})\frac{N(N^2-1)}{6}}$$

the "vacuum charge" $\qquad (\hat{m} \equiv m - \frac{1}{2})$

degeneracy: 1, of course…

- k=1: $I_{k=0}\left(Ne^{-\beta}e^{\beta\hat{m}} - (N-1)e^{-\beta}e^{\beta\hat{m}}\right) = I_{k=0} \cdot e^{-\beta}e^{\beta\hat{m}}$

Keep s, excite k_{SD} = 1. Keep k_{SD} = 0, decrease s in many ways

completely agrees with SUGRA index on $AdS_7 \times S^4$

- k=2: N=1 already discussed. For N≥2, one obtains $I_{k=0}$ times

Contributions from various anti-self-dual fluxes

$$\left[\begin{array}{l} q^2\left[\frac{N(N+1)}{2}y^2 + Ny(y_1+y_2+y_3) - N\left(y_1^{-1}+y_2^{-1}+y_3^{-1}\right) + Ny^{-1}\right] \\ -(N-1)(N-2)q^2y^2 - (N-1)q^2\left[y^2 + y(y_1+y_2+y_3) - (y_1^{-1}+y_2^{-1}+y_3^{-1}) + y^{-1}\right] \\ +\frac{(N-2)(N-3)}{2}q^2y^2 \end{array}\right. = q^2\left[2y^2 + y(y_1+y_2+y_3) - (y_1^{-1}+y_2^{-1}+y_3^{-1}) + y^{-1}\right]$$

- Spectrum doesn't depend on N for N≥k: familiar behavior for CFT with AdS dual.

(Based on this fact, we made similar studies at k=3 from U(3) index: agrees with SUGRA dual)

Concluding remarks

- From $Z[CP^2 \times S^1]$, one can study the BPS spectrum of 6d SCFT.

- We can learn more about the 6d (2,0) dynamics from this observable.

- E.g. finite N, large E: dependence on N for k>N provides information on "trace relations" of gauge theory: what is the gauge theory structure of 6d (2,0) CFT?
- We made some preliminary study on this high E, finite N indices: N=2, k=3.

- $CP^2 \times R$ approach will presumably extend easily to 6d (1,0) SCFT's.

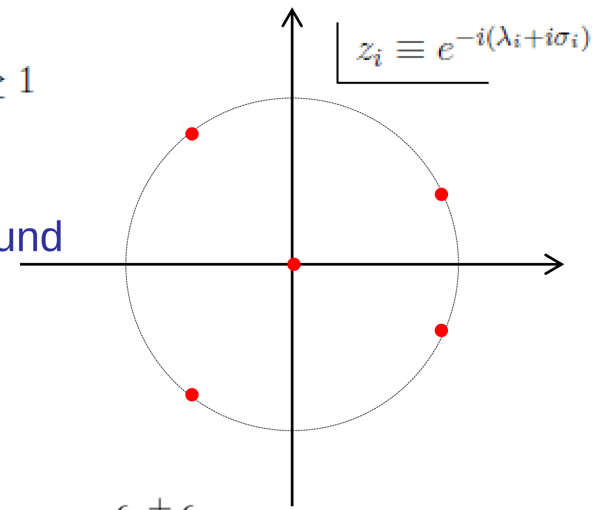
- Other 6-manifolds (and 5d SYM approaches)?
- E.g. $S^4 \times T^2$ from $S^4 \times S^1$...?
- 5d SYM with boundary…? (to study more general 6-manifolds from reduction to 5d)

Supplementary slide: integration contour

- Integration variables are essentially holonomies: naively, the contour is a circle.

(It is convenient to take imaginary U(1)^2 chemical potentials, to clearly see the situation.)

The naïve contour can hit poles at $\frac{z_i}{z_j} = e^{\pm\beta(p\epsilon_1 + q\epsilon_2)}$ with $p \geq 1$, $q \geq 1$



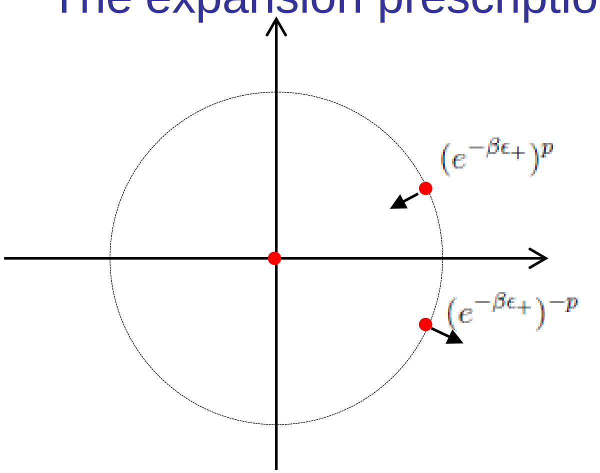
- unambiguous way to go around them: charge unitarity bound

$$\{Q^{\pm\pm}_{\mp\mp\mp}, S^{\mp\mp}_{\pm\pm\pm}\} \geq 0 \ \rightarrow \ E_{BPS} \geq 2(\pm R_1 \pm R_2) \pm j_1 \pm j_2 \pm j_3$$

$$R_1 \geq 0\ ,\quad R_2 \geq 0\ ,\quad j_1 + j_2 \geq 0\ ,\quad j_2 + j_3 \geq 0\ ,\quad j_3 + j_1 \geq 0$$

- Expand all $Z^{(i)}_{pert}$ $Z^{(i)}_{inst}$ in positive powers of $e^{-\beta\epsilon_+} = e^{\frac{3\beta a_i}{2}}$. $\epsilon_\pm = \frac{\epsilon_1 \pm \epsilon_2}{2}$

- The expansion prescription implies $\epsilon_+ \to i\,\mathrm{Im}(\epsilon_+) + \varepsilon$



So expansion prescriptions require the following contour deformation.