Painlevé functions and conformal blocks

Oleg Lisovyy

LMPT, Tours, France & BITP, Kyiv, Ukraine

with O. Gamayun, N. Iorgov and J. Teschner

Painlevé equations:

- classification of ODEs $w'' = F(w, w', t)$ without movable critical points
- non-autonomous hamiltonian systems
- confluence cascade

$$PVI \to PV \nearrow PIV \searrow PII \to PI$$
$$PVI \to PV \searrow PIII \nearrow PII \to PI$$

Solutions:

- classical special functions

$${}_2F_1 \to \textit{Whittaker} \nearrow \textit{Hermite} \searrow \textit{Airy} \to \varnothing$$
$${}_2F_1 \to \textit{Whittaker} \searrow \textit{Bessel} \nearrow \textit{Airy} \to \varnothing$$

- elliptic (PVI)
- algebraic
- transcendental (almost all solutions!)

Example: 2D Ising model

Diagonal two-point correlation function

$$D_N = \langle \sigma(0,0)\sigma(N,N)\rangle_{T<T_c}$$

is a Painlevé VI τ-function [Jimbo, Miwa, '81]:

$$D_N = (1-t)^{\frac{N^2}{2}}\,\tau(t)$$

More precisely, $\sigma(t) = t(t-1)\dfrac{d}{dt}\ln\tau$ satisfies

$$-\frac{1}{2}\Big(t(t-1)\sigma''\Big)^2 =$$
$$= \det\begin{pmatrix} 2\theta_0^2 & t\sigma'-\sigma & \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\sigma'-\sigma & 2\theta_t^2 & (t-1)\sigma'-\sigma \\ \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\sigma'-\sigma & 2\theta_1^2 \end{pmatrix}$$

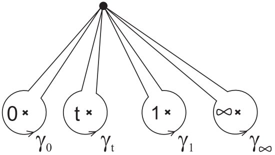
- temperature parameter $t = (\sinh 2\mathcal{K}_x \sinh 2\mathcal{K}_y)^{-2}$
- $\boldsymbol{\theta} = \left(0, \frac{N}{2}, \frac{N}{2}, \frac{1}{2}\right)$
- special case of hypergeometric solutions
- transcendental solutions of PV/PIII in the scaling limit [McCoy, Tracy, Wu, Barouch '76]

Painlevé VI and isomonodromy

PVI describes monodromy preserving deformations of rank 2 linear systems on $\mathbb{P}^1$ with 4 regular singular points $0, t, 1, \infty$:

$$\frac{d\Phi}{dz} = \mathcal{A}(z)\Phi, \qquad \mathcal{A}(z) = \frac{\mathcal{A}_0}{z} + \frac{\mathcal{A}_t}{z-t} + \frac{\mathcal{A}_1}{z-1}$$

- matrices $\mathcal{A}_\nu$ are 2×2, traceless, with eigenvalues $\pm\theta_\nu$
- $\mathcal{A}_0 + \mathcal{A}_t + \mathcal{A}_1 \overset{def}{=} -\mathcal{A}_\infty = \mathrm{diag}\left\{-\theta_\infty, \theta_\infty\right\}$
- 3 monodromy matrices $\mathcal{M}_{0,t,1} \in G = SL(2,\mathbb{C})$ (note $\mathcal{M}_\infty\mathcal{M}_1\mathcal{M}_t\mathcal{M}_0 = \mathbf{1}$)
- monodromy manifold $\mathcal{M} = G^3/G$, $\dim\mathcal{M} = 6$



Painlevé VI and isomonodromy (continued)

Schlesinger equations:

$$\frac{d\mathcal{A}_0}{dt} = \frac{[\mathcal{A}_t, \mathcal{A}_0]}{t}, \qquad \frac{d\mathcal{A}_1}{dt} = \frac{[\mathcal{A}_t, \mathcal{A}_1]}{t-1}$$

- Lax form $\Rightarrow \theta_{0,t,1,\infty}$ are conserved
- remains 2 degrees of freedom (recall that $\mathcal{A}_0 + \mathcal{A}_t + \mathcal{A}_1 = -\mathcal{A}_\infty$)
- $\left(\frac{\mathcal{A}_0}{z} + \frac{\mathcal{A}_t}{z-t} + \frac{\mathcal{A}_1}{z-1}\right)_{12} = \frac{k(t)(z-w(t))}{z(z-t)(z-1)} \Rightarrow$ standard form of PVI for $w(t)$
- $\sigma = (t-1)\,\mathrm{tr}\,\mathcal{A}_0\mathcal{A}_t + t\,\mathrm{tr}\,\mathcal{A}_1\mathcal{A}_t \Rightarrow$ sigma form of PVI for $\sigma(t)$

Painlevé VI and isomonodromy (continued)

Sigma Painlevé VI:

$$\left(t(t-1)\sigma''\right)^2=-2\det\begin{pmatrix} 2\theta_0^2 & t\sigma'-\sigma & \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\sigma'-\sigma & 2\theta_t^2 & (t-1)\sigma'-\sigma \\ \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\sigma'-\sigma & 2\theta_1^2 \end{pmatrix}$$

Monodromy data:

- to any solution corresponds (the conjugacy class of) a triple $(\mathcal{M}_0,\mathcal{M}_t,\mathcal{M}_1)$
- $p_\nu=2\cos 2\pi\theta_\nu=\operatorname{tr}\mathcal{M}_\nu$ (with $\nu=0,t,1,\infty$) give four PVI parameters
- remaining two coordinates $\Rightarrow$ integration constants
- introduce $p_{\mu\nu}=2\cos 2\pi\sigma_{\mu\nu}=\operatorname{tr}\mathcal{M}_\mu\mathcal{M}_\nu$, then [Jimbo, '82]

$$p_{0t}p_{1t}p_{01}+p_{0t}^2+p_{1t}^2+p_{01}^2+p_0^2+p_t^2+p_1^2+p_\infty^2+p_0p_tp_1p_\infty=$$
$$=\left(p_0p_t+p_1p_\infty\right)p_{0t}+\left(p_1p_t+p_0p_\infty\right)p_{1t}+\left(p_0p_1+p_tp_\infty\right)p_{01}+4$$

The triple $\boldsymbol{\sigma}$ satisfying the above relation can be interpreted as a pair of PVI integration constants. Our task is: given $\boldsymbol{\sigma}$, to obtain the corresponding solution.

Jimbo's formula ['82]

- expresses the asymptotics of $\tau(t)$ as $t \to 0$, 1, or ∞ in terms of monodromy
- e.g. for $t \to 0$, denote $\sigma = \sigma_{0t}$ and choose $0 < |\mathrm{Re}\,\sigma| < \frac{1}{2}$
- also denote $\Delta_\nu = \theta_\nu^2$ $(\nu = 0, t, 1, \infty)$ and $\Delta_\sigma = \sigma^2$; then

$$\tau(t) = \mathrm{const} \cdot \left(t^{\Delta_\sigma - \Delta_0 - \Delta_t} + C_{\pm 1} t^{\Delta_\sigma \pm 1 - \Delta_0 - \Delta_t} + \text{ smaller terms } \right),$$

with

$$C_{\pm 1} = \frac{\Gamma^2(1 \mp 2\sigma)}{\Gamma^2(1 \pm 2\sigma)} \prod_{\epsilon = \pm} \frac{\Gamma(1 + \epsilon\theta_0 + \theta_t \pm \sigma)\,\Gamma(1 + \epsilon\theta_\infty + \theta_1 \pm \sigma)}{\Gamma(1 + \epsilon\theta_0 + \theta_t \mp \sigma)\,\Gamma(1 + \epsilon\theta_\infty + \theta_1 \mp \sigma)} \times$$
$$\times \frac{\left(\theta_0^2 - (\theta_t \mp \sigma)^2\right)\left(\theta_\infty^2 - (\theta_1 \mp \sigma)^2\right)}{4\sigma^2 (1 \pm 2\sigma)^2} (-s_{0t})^{\pm 1},$$

and

$$s_{0t}^{\pm 1} \left(\cos 2\pi(\theta_t \mp \sigma) - \cos 2\pi\theta_0\right)\left(\cos 2\pi(\theta_1 \mp \sigma) - \cos 2\pi\theta_\infty\right) =$$
$$= \left(\cos 2\pi\theta_t \cos 2\pi\theta_1 + \cos 2\pi\theta_0 \cos 2\pi\theta_\infty \pm i \sin 2\pi\sigma \cos 2\pi\sigma_{01}\right) -$$
$$- \left(\cos 2\pi\theta_0 \cos 2\pi\theta_1 + \cos 2\pi\theta_t \cos 2\pi\theta_\infty \mp i \sin 2\pi\sigma \cos 2\pi\sigma_{1t}\right) e^{\pm 2\pi i \sigma}.$$

- higher-order corrections can be determined recursively from σPVI

Higher order corrections

$$\tau(t) \sim t^{\Delta_\sigma - \Delta_0 - \Delta_t}\Big(1 + \mathcal{B}_1(\boldsymbol{\theta}, \sigma)t + \qquad \ldots\Big)$$

$$+C_{\pm 1}\; t^{\Delta_{\sigma\pm 1} - \Delta_0 - \Delta_t}$$

with

$$\mathcal{B}_1(\boldsymbol{\theta}, \sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_\infty + \Delta_1)}{2\Delta_\sigma},$$

Higher order corrections

$$\tau(t) \sim t^{\Delta_\sigma-\Delta_0-\Delta_t}\left(1+\mathcal{B}_1(\boldsymbol{\theta},\sigma)t+\mathcal{B}_2(\boldsymbol{\theta},\sigma)t^2\ldots\right)+$$
$$+C_{\pm1}\, t^{\Delta_{\sigma\pm1}-\Delta_0-\Delta_t}\left(1+\mathcal{B}_1^{(\pm1)}(\boldsymbol{\theta},\sigma)t+\ldots\right)+C_{\pm2}t^{\Delta_{\sigma\pm2}-\Delta_0-\Delta_t}\left(1+\ldots\right)$$

with

$$\mathcal{B}_1(\boldsymbol{\theta},\sigma)=\frac{(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_\infty+\Delta_1)}{2\Delta_\sigma},$$

$$\mathcal{B}_2(\boldsymbol{\theta},\sigma)=\frac{(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_0+\Delta_t+1)(\Delta_\sigma-\Delta_\infty+\Delta_1)(\Delta_\sigma-\Delta_\infty+\Delta_1+1)}{4\Delta_\sigma(2\Delta_\sigma+1)}$$
$$+\frac{\left[(1+2\Delta_\sigma)\,(\Delta_0+\Delta_t)+\Delta_\sigma(\Delta_\sigma+1)-3(\Delta_0-\Delta_t)^2\right]\left[(1+2\Delta_\sigma)\,(\Delta_\infty+\Delta_1)+\Delta_\sigma(\Delta_\sigma+1)-3(\Delta_\infty-\Delta_1)^2\right]}{2\,(2\Delta_\sigma+1)\,(4\Delta_\sigma-1)^2},$$

$$\mathcal{B}_1^{(\pm1)}(\boldsymbol{\theta},\sigma)=\mathcal{B}_1(\boldsymbol{\theta},\sigma\pm1).$$

Higher order corrections

$$\tau(t) \sim t^{\Delta_\sigma-\Delta_0-\Delta_t}\Big(1+\mathcal{B}_1(\boldsymbol{\theta},\sigma)t+\mathcal{B}_2(\boldsymbol{\theta},\sigma)t^2\ldots\Big)+$$
$$+C_{\pm 1}\, t^{\Delta_{\sigma\pm 1}-\Delta_0-\Delta_t}\Big(1+\mathcal{B}_1^{(\pm 1)}(\boldsymbol{\theta},\sigma)t+\ldots\Big)+C_{\pm 2}t^{\Delta_{\sigma\pm 2}-\Delta_0-\Delta_t}\Big(1+\ldots\Big)$$

with

$$\mathcal{B}_1(\boldsymbol{\theta},\sigma)=\frac{(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_\infty+\Delta_1)}{2\Delta_\sigma},$$

$$\mathcal{B}_2(\boldsymbol{\theta},\sigma)=\frac{(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_0+\Delta_t+1)(\Delta_\sigma-\Delta_\infty+\Delta_1)(\Delta_\sigma-\Delta_\infty+\Delta_1+1)}{4\Delta_\sigma(2\Delta_\sigma+1)}$$
$$+\frac{\left[(1+2\Delta_\sigma)(\Delta_0+\Delta_t)+\Delta_\sigma(\Delta_\sigma+1)-3(\Delta_0-\Delta_t)^2\right]\left[(1+2\Delta_\sigma)(\Delta_\infty+\Delta_1)+\Delta_\sigma(\Delta_\sigma+1)-3(\Delta_\infty-\Delta_1)^2\right]}{2(2\Delta_\sigma+1)(4\Delta_\sigma-1)^2+2(c-1)(2\Delta_\sigma+1)^2},$$

$$\mathcal{B}_1^{(\pm 1)}(\boldsymbol{\theta},\sigma)=\mathcal{B}_1(\boldsymbol{\theta},\sigma\pm 1).$$

Higher order corrections

$$\tau(t) \sim t^{\Delta_\sigma - \Delta_0 - \Delta_t}\Big(1 + \mathcal{B}_1(\boldsymbol{\theta},\sigma)t + \mathcal{B}_2(\boldsymbol{\theta},\sigma)t^2 \ldots\Big) +$$

$$+ C_{\pm 1}\, t^{\Delta_{\sigma\pm1} - \Delta_0 - \Delta_t}\Big(1 + \mathcal{B}_1^{(\pm1)}(\boldsymbol{\theta},\sigma)t + \ldots\Big) + C_{\pm 2}t^{\Delta_{\sigma\pm2} - \Delta_0 - \Delta_t}\Big(1 + \ldots\Big)$$

with

$$\mathcal{B}_1(\boldsymbol{\theta},\sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_\infty + \Delta_1)}{2\Delta_\sigma},$$

$$\mathcal{B}_2(\boldsymbol{\theta},\sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_0 + \Delta_t + 1)(\Delta_\sigma - \Delta_\infty + \Delta_1)(\Delta_\sigma - \Delta_\infty + \Delta_1 + 1)}{4\Delta_\sigma(2\Delta_\sigma + 1)}$$

$$+ \frac{\left[(1+2\Delta_\sigma)(\Delta_0 + \Delta_t) + \Delta_\sigma(\Delta_\sigma + 1) - 3(\Delta_0 - \Delta_t)^2\right]\left[(1+2\Delta_\sigma)(\Delta_\infty + \Delta_1) + \Delta_\sigma(\Delta_\sigma + 1) - 3(\Delta_\infty - \Delta_1)^2\right]}{2(2\Delta_\sigma + 1)(4\Delta_\sigma - 1)^2 + 2(c-1)(2\Delta_\sigma + 1)^2},$$

$$\mathcal{B}_1^{(\pm1)}(\boldsymbol{\theta},\sigma) = \mathcal{B}_1(\boldsymbol{\theta},\sigma \pm 1).$$

Observation. PVI tau function is a linear combination of $c = 1$ conformal blocks:

$$\tau(t) = \sum_{n\in\mathbb{Z}} C_n\,\mathcal{B}(\boldsymbol{\theta},\sigma + n, t)$$

Conformal blocks

- Virasoro algebra:

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{c}{12}\, n\left(n^2-1\right)\delta_{m+n,0}$$

- Verma module $\mathcal{V}_\Delta$ generated by $|\Delta\rangle$ s.t.

$$L_0|\Delta\rangle = \Delta|\Delta\rangle, \qquad L_{>0}|\Delta\rangle = 0$$

- Descendant states are obtained by the action of raising operators

$$\mathcal{L}_{-\lambda}|\Delta\rangle = L_{-\lambda_N}\dots L_{-\lambda_1}|\Delta\rangle, \qquad \lambda \in \mathbb{Y}$$

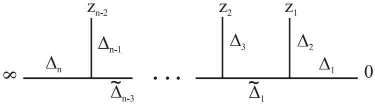
$$\text{e.g.}\quad \mathcal{L}_{-\,\yng(4,2,2,1)}|\Delta\rangle = L_{-1}L_{-2}^2L_{-4}|\Delta\rangle$$

- Chiral vertex operators $V^{\Delta}_{\Delta_2,\Delta_1}(z) : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ are uniquely defined by

$$\left[L_n, V^{\Delta}_{\Delta_2,\Delta_1}(z)\right] = z^n\left\{z\partial_z + (n+1)\,\Delta\right\} V^{\Delta}_{\Delta_2,\Delta_1}(z)$$

- Conformal blocks can be defined as matrix elements

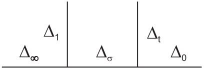
$$\langle\Delta_n|V^{\Delta_{n-1}}_{\Delta_n,\tilde{\Delta}_{n-3}}(z_{n-2})\dots V^{\Delta_3}_{\tilde{\Delta}_2,\tilde{\Delta}_1}(z_2)V^{\Delta_2}_{\tilde{\Delta}_1,\Delta_1}(z_1)|\Delta_1\rangle$$



Computation of conformal blocks

Painlevé VI corresponds to

$$\mathcal{B}(t) = \langle \Delta_\infty | V^{\Delta_1}_{\Delta_\infty, \Delta_\sigma}(1) V^{\Delta_t}_{\Delta_\sigma, \Delta_0}(t) | \Delta_0 \rangle =$$



1. direct (inversion of Kac-Shapovalov matrix)

$$\mathcal{B}(t) = t^{\Delta - \Delta_0 - \Delta_t} \sum_{\lambda, \mu \in \mathbb{Y}} \gamma_\lambda(\Delta, \Delta_1, \Delta_\infty) \left[Q(\Delta) \right]^{-1}_{\lambda\mu} \gamma_\mu(\Delta, \Delta_t, \Delta_0)\, t^{|\lambda|}$$

Here

$$\gamma_\mu(\Delta, \Delta_t, \Delta_0) = \prod_{j=1}^{\ell(\mu)} \left(\Delta - \Delta_0 + \mu_j \Delta_t + \sum_{k=1}^{j-1} \mu_k \right)$$

and $Q_{\lambda\mu}(\Delta) = \langle \Delta | \mathcal{L}_\lambda \mathcal{L}_{-\mu} | \Delta \rangle$, so that e.g

$$Q_{(3,1)\,(2,2)}(\Delta) = \langle \Delta | L_3 L_1 L_{-2}^2 | \Delta \rangle$$

2. recursion relation [Zamolodchikov, '84]
3. AGT correspondence [Alday, Gaiotto, Tachikawa, '09]:

$$\mathcal{B}(t) = \mathcal{Z}_{\text{inst}}(t) = \begin{array}{c}\text{combinatorial sum}\\ \text{over tuples of partitions}\end{array}$$

Proved in [Alba, Fateev, Litvinov, Tarnopolsky, '10]

Structure constants

Jimbo's asymptotic formula can be interpreted as a recurrence relation

$$\frac{C_{n\pm1}}{C_n}=\frac{\Gamma^2\left(1\mp 2(\sigma_{0t}+n)\right)}{\Gamma^2\left(1\pm 2(\sigma_{0t}+n)\right)}\prod_{\epsilon=\pm}\frac{\Gamma\left(1+\epsilon\theta_0+\theta_t\pm(\sigma_{0t}+n)\right)\Gamma\left(1+\epsilon\theta_\infty+\theta_1\pm(\sigma_{0t}+n)\right)}{\Gamma\left(1+\epsilon\theta_0+\theta_t\mp(\sigma_{0t}+n)\right)\Gamma\left(1+\epsilon\theta_\infty+\theta_1\mp(\sigma_{0t}+n)\right)}\times$$
$$\times\frac{\left(\theta_0^2-(\theta_t\mp(\sigma_{0t}+n))^2\right)\left(\theta_\infty^2-(\theta_1\mp(\sigma_{0t}+n))^2\right)}{4\left(\sigma_{0t}+n\right)^2\left(1\pm 2(\sigma_{0t}+n)\right)^2}\left(-s_{0t}\right)^{\pm1}$$

with the solution in terms of Barnes functions

$$C_n\left(\boldsymbol{\theta},\boldsymbol{\sigma}\right)=s_{0t}^n\frac{\prod_{\epsilon,\epsilon'=\pm}G\left(1+\theta_t+\epsilon\theta_0+\epsilon'(\sigma_{0t}+n)\right)G\left(1+\theta_1+\epsilon\theta_\infty+\epsilon'(\sigma_{0t}+n)\right)}{G\left(1+2(\sigma_{0t}+n)\right)G\left(1-2(\sigma_{0t}+n)\right)}$$

Recursion relation for Barnes G-function: $G(z+1)=\Gamma(z)G(z)$

Structure constants (continued)

Changing normalization of the vertex operators

$$V_{\theta_3,\theta_1}^{\theta_2}(z)\,|\,\theta_1\rangle = N(\theta_3,\theta_2,\theta_1)\,z^{\theta_3^2-\theta_1^2-\theta_2^2}\Big[\,|\,\theta_3\rangle \,+\, O(z)\Big]$$

from $N(\theta_3,\theta_2,\theta_1)=1$ to

$$N(\theta_3,\theta_2,\theta_1)=\frac{\prod_{\epsilon=\pm} G(1+\theta_3+\epsilon(\theta_1+\theta_2))\,G(1-\theta_3+\epsilon(\theta_1-\theta_2))}{G(1-2\theta_1)G(1-2\theta_2)G(1+2\theta_3)},$$

Painlevé VI tau function becomes Fourier transform of $c=1$ conformal block:

$$\boxed{\tau(t)=\sum_{n\in\mathbb{Z}} e^{in\tau}\,\mathcal{B}(\boldsymbol{\theta},\sigma+n,t)}$$

- 4 parameters $\boldsymbol{\theta} \Longleftrightarrow$ external momenta
- 2 integration constants $(\sigma,\tau) \Longleftrightarrow$ intermediate momentum + generating parameter

Claim

Complete expansion of Painlevé VI tau function near $t = 0$ can be written as

$$\tau(t) = \text{const} \cdot \sum_{n\in\mathbb{Z}} C_n(\boldsymbol{\theta}, \boldsymbol{\sigma})\, t^{(\sigma_{0t}+n)^2-\theta_0^2-\theta_t^2} \mathcal{B}(\boldsymbol{\theta}, \sigma_{0t}+n; t).$$

The function $\mathcal{B}(\boldsymbol{\theta}, \sigma; t)$ is a power series in t which coincides with the general $c = 1$ conformal block and is explicitly given by

$$\mathcal{B}(\boldsymbol{\theta}, \sigma; t) = t^{\sigma^2-\theta_0^2-\theta_t^2}(1-t)^{2\theta_t\theta_1} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}(\boldsymbol{\theta}, \sigma)\, t^{|\lambda|+|\mu|},$$

$$\mathcal{B}_{\lambda,\mu}(\boldsymbol{\theta}, \sigma) = \prod_{(i,j)\in\lambda} \frac{\left((\theta_t+\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1+\sigma+i-j)^2-\theta_\infty^2\right)}{h_\lambda^2(i,j)\left(\lambda'_j-i+\mu_i-j+1+2\sigma\right)^2} \times$$

$$\times \prod_{(i,j)\in\mu} \frac{\left((\theta_t-\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1-\sigma+i-j)^2-\theta_\infty^2\right)}{h_\mu^2(i,j)\left(\mu'_j-i+\lambda_i-j+1-2\sigma\right)^2}.$$

The structure constants $\{C_n(\boldsymbol{\theta}, \boldsymbol{\sigma})\}_{n\in\mathbb{Z}}$ can be written in terms of Barnes G-function,

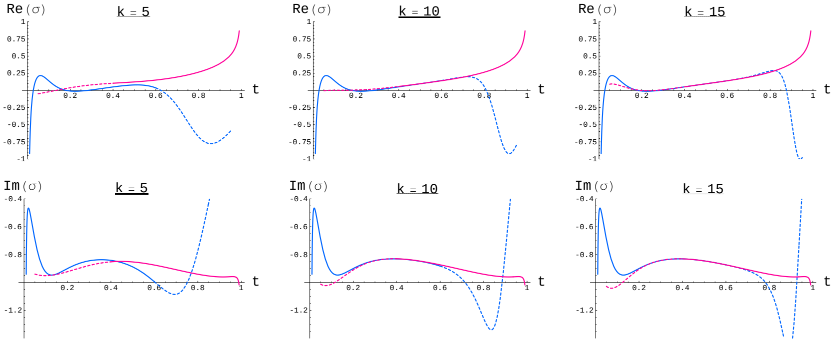
$$C_n(\boldsymbol{\theta}, \boldsymbol{\sigma}) = s_{0t}^n \frac{\prod_{\epsilon,\epsilon'=\pm} G\left(1+\theta_t+\epsilon\theta_0+\epsilon'(\sigma_{0t}+n)\right) G\left(1+\theta_1+\epsilon\theta_\infty+\epsilon'(\sigma_{0t}+n)\right)}{G\left(1+2(\sigma_{0t}+n)\right) G\left(1-2(\sigma_{0t}+n)\right)}$$

Remarks

- checked about 30 first terms in the asymptotic expansion of τ (up to level 10, ~ 500 bipartitions) in full generality
- to prove rigorously, it is sufficient to demonstrate two bilinear relations satisfied by $c = 1$ conformal blocks
- expansions at 1, ∞ are obtained by parameter change; for example, near $t = 1$

$$\theta_0 \leftrightarrow \theta_1, \quad \sigma_{0t} \leftrightarrow \sigma_{1t}, \qquad p'_{01} = \omega_{01} - p_{01} - p_{0t}p_{1t}.$$

- series representations suitable for numerical evaluation of PVI functions



$$\begin{pmatrix} \theta_0 \\ \theta_t \\ \theta_1 \\ \theta_\infty \end{pmatrix} = \begin{pmatrix} 0.501790 + 0.216884i \\ 0.382251 + 0.723641i \\ 0.152700 + 0.358959i \\ 0.158518 + 0.674992i \end{pmatrix}, \quad \begin{pmatrix} \sigma_{0t} \\ \sigma_{1t} \end{pmatrix} = \begin{pmatrix} 0.837497 + 0.943080i \\ 0.411398 + 0.480375i \end{pmatrix}$$

Coalescence diagram revisited

$$
\begin{array}{ccccccccc}
\mathrm{VI} & \longrightarrow & \mathrm{V} & \longrightarrow & \mathrm{III}_1 & \longrightarrow & \mathrm{III}_2 & \longrightarrow & \mathrm{III}_3 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{IV} & \longrightarrow & \mathrm{II} & \longrightarrow & \mathrm{I} & &
\end{array}
$$

- can easily write similar expansions for Painlevé V and $\mathrm{III}_{1,2,3}$
- coalescence corresponds to decoupling of matter hypermultiplets

$$
\begin{array}{c} N_f = 4 \\ (P_{\mathrm{VI}}) \end{array} \xrightarrow{\mu_4 \to \infty} \begin{array}{c} N_f = 3 \\ (P_{\mathrm{V}}) \end{array} \xrightarrow{\mu_3 \to \infty} \begin{array}{c} N_f = 2 \\ (P_{\mathrm{III}_1}) \end{array} \xrightarrow{\mu_2 \to \infty} \begin{array}{c} N_f = 1 \\ (P_{\mathrm{III}_2}) \end{array} \xrightarrow{\mu_1 \to \infty} \begin{array}{c} \text{pure gauge theory} \\ (P_{\mathrm{III}_3}) \end{array}
$$

Painlevé III$_3$:

Equivalent to the radial sinh-Gordon

$$\psi'' + \frac{1}{r}\psi' = \frac{1}{2}\sinh 2\psi.$$

Equation for the tau function:

$$\left(D^4 + (1-2\delta)D^2 + 4t\right)\tau\cdot\tau = 0.$$

where $\delta = t\dfrac{d}{dt}$ and D is the associated Hirota derivative

Conformal expansion:

$$\tau(t) = \sum_{n\in\mathbb{Z}} C_{\sigma+n}\, s^n\, \mathcal{B}_{\sigma+n}(t)\,,$$

- integration constants σ, s
- $C_\sigma = [G(1+2\sigma)G(1-2\sigma)]^{-1}$
- AGT representation:

$$\mathcal{B}_\sigma(t) = \sum_{\lambda,\mu\in\mathbb{Y}} t^{\sigma^2+|\lambda|+|\mu|}\left[\prod_{(i,j)\in\lambda} h_\lambda(i,j)\left(\lambda'_j+\mu_i-i-j+1+2\sigma\right)\times\right.$$
$$\left.\times\prod_{(i,j)\in\mu} h_\mu(i,j)\left(\lambda_i+\mu'_j-i-j+1-2\sigma\right)\right]^{-2}$$

Painlevé III$_3$ (continued):

It suffices to prove

$$\sum_{n\in\mathbb{Z}} \chi_{\sigma,n} \left(D^4 + (1-2\delta)D^2 + 4t\right) \mathcal{B}_{\sigma+n} \cdot \mathcal{B}_{\sigma-n} = 0$$

with

$$\chi_{\sigma,n} = \prod_{k=1-2n}^{2n-1} (2\sigma - k)^{-2(2n-|k|)}$$

and the same relation with $n \in \mathbb{Z} + \frac{1}{2}$.

Algebraic formulation:

- Consider the sequence of states $|n\rangle$ with $n = 0, 1, \ldots$ such that

$$|0\rangle = |\Delta\rangle, \qquad L_1|n\rangle = |n-1\rangle, \qquad L_2|n\rangle = 0.$$

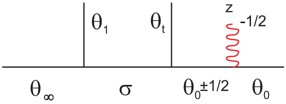
- Whittaker vector $|\Delta\rangle_W = \sum_{n=0}^{\infty} t^{\frac{n+\Delta}{2}} |n\rangle$ satisfies

$$L_0|\Delta\rangle_W = 2t\frac{d}{dt}|\Delta\rangle_W, \qquad L_1|\Delta\rangle_W = \sqrt{t}\,|\Delta\rangle_W, \qquad L_2|\Delta\rangle_W = 0$$

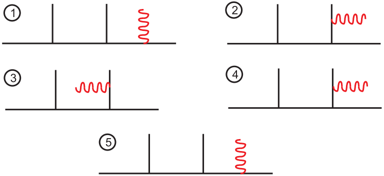
- irregular conformal block $\mathcal{B}_\sigma(t) = {}_W\langle\Delta|\Delta\rangle_W$ with $\Delta = \sigma^2$

Proof idea:

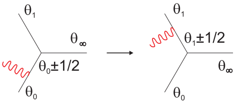
- Instead of 4-point conformal blocks for PVI consider 5-point conformal blocks with level 2 degenerate insertions



- analytic continuation in z can be realized as a sequence of simple moves

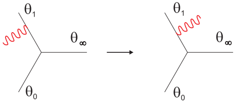


- Each of the moves produces a 2×2 operator-valued monodromy matrix:
  - S-move:



$$S_{\epsilon'\epsilon}\left(\theta_0,\theta_1,\theta_\infty\right)=\epsilon'\,\frac{\cos\pi\left(\theta_\infty-\epsilon'\theta_0+\epsilon\theta_1\right)}{\sin 2\pi\theta_1}$$

  - B-move:



$$B\left(\theta_1\right)=\begin{pmatrix} e^{-i\pi\theta_1} & 0 \\ 0 & e^{i\pi\theta_1}\end{pmatrix}$$

  - T-move:

$$T\left(\sigma\right)=\begin{pmatrix} 0 & \nabla_{-1/2} \\ \nabla_{1/2} & 0\end{pmatrix}$$

- entries can shift intermediate momentum σ by half-integers
- diagonalizing by Fourier transform, we obtain the solution of the isomonodromic Riemann-Hilbert problem, from which one can extract the tau function

Main claim:

Let $C_{0,n}$ denote $\mathbb{P}^1 \backslash \{a_1 = 0, a_2 \ldots, a_{n-1}, a_n = \infty\}$. To each of $n-3$ closed curves γ_r decomposing $C_{0,n}$ into $n-2$ pairs of pants we assign a pair (σ_r, τ_r) and consider the associated $c = 1$ conformal block.

Fourier transforms

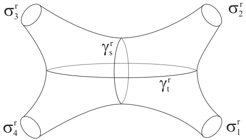
$$\Phi_{\epsilon'\epsilon}(z) = \frac{\sum_{\mathbf{n}\in\mathbb{Z}^n} \left\langle \theta_n - \frac{\epsilon'}{2} \middle| \psi_\epsilon(z)\, V^{\theta_{n-1}}_{\theta_n + \frac{\epsilon-\epsilon'}{2}, \sigma_{n-3}}(a_{n-1}) \ldots V^{\theta_2}_{\sigma_1,\theta_1}(a_2) \middle| \theta_1 \right\rangle e^{i\mathbf{n}\cdot\boldsymbol{\tau}}}{\tau(\mathbf{a}|\boldsymbol{\theta}|\boldsymbol{\sigma},\boldsymbol{\tau})},$$

$$\tau(\mathbf{a}|\boldsymbol{\theta}|\boldsymbol{\sigma},\boldsymbol{\tau}) = \sum_{\mathbf{n}\in\mathbb{Z}^n} \left\langle \theta_n \middle| \psi_s(z)\, V^{\theta_{n-1}}_{\theta_n,\sigma_{n-3}}(a_{n-1}) \ldots V^{\theta_2}_{\sigma_1,\theta_1}(a_2) \middle| \theta_1 \right\rangle e^{i\mathbf{n}\cdot\boldsymbol{\tau}}$$

provide, respectively, the solution of the general isomonodromic Riemann-Hilbert problem for rank 2 Fuchsian systems and the associated tau function.

- $\psi_\epsilon(z) \equiv V^{-1/2}_{p-\epsilon/2,p}(z)$
- $\boldsymbol{\sigma}, \boldsymbol{\tau}$ are local Darboux coordinates on the $(2n-6)$-dimensional moduli space $\mathrm{Hom}(\pi_1(C_{0,n}), SL(2,\mathbb{C}))/\sim$ with fixed local monodromy exponents

Trace functions:



$$L_s^r = 2\cos(2\pi\sigma_r)\,,$$
$$(\sin(2\pi\sigma_r))^2\, L_t^r = C_+(\sigma_r)\, e^{i\tau_r} + C_0(\sigma_r) + C_-(\sigma_r)\, e^{-i\tau_r}\,,$$

where

$$C_+(\sigma_r) = -4\prod_{\epsilon=\pm} \sin\pi(\sigma_r + \epsilon(\sigma_1^r - \sigma_2^r))\sin\pi(\sigma_r + \epsilon(\sigma_3^r - \sigma_4^r))$$
$$C_0(\sigma_r) = 2\cos(2\pi\sigma_r)\left[\cos(2\pi\sigma_1^r)\cos(2\pi\sigma_3^r) + \cos(2\pi\sigma_2^r)\cos(2\pi\sigma_4^r)\right] - 2(\cos(2\pi\sigma_2^r)\cos(2\pi\sigma_3^r) + \cos(2\pi\sigma_1^r)\cos(2\pi\sigma_4^r))\,.$$
$$C_-(\sigma_r) = -4\prod_{\epsilon=\pm} \sin\pi(\sigma_r + \epsilon(\sigma_1^r + \sigma_2^r))\sin\pi(\sigma_r + \epsilon(\sigma_3^r + \sigma_4^r))\,.$$

- in particular, this proves Jimbo asymptotic formula for Painlevé VI

Riccati solutions

- parameters satisfy

$$\begin{cases} \omega_{0t} = 2p_{0t} + p_{1t}p_{01}, \\ \omega_{1t} = 2p_{1t} + p_{0t}p_{01}, \\ \omega_{01} = 2p_{01} + p_{0t}p_{1t}. \end{cases}$$

- simplest case $\theta_0 + \theta_t + \theta_1 + \theta_\infty = 0$, $\boldsymbol{\sigma} = (\theta_0 + \theta_t, \theta_1 + \theta_t, \theta_0 + \theta_1)$:

$$\tau(t) = \mathrm{const} \cdot t^{2\theta_0\theta_t}(1-t)^{2\theta_t\theta_1}.$$

- four-point correlator $\langle \mathcal{V}_{\theta_0}(0)\mathcal{V}_{\theta_t}(t)\mathcal{V}_{\theta_1}(1)\mathcal{V}_{\theta_\infty}(\infty)\rangle$ of chiral vertex operators $\mathcal{V}_\theta(z) = :e^{i\sqrt{2}\theta\phi(z)}:$ (only one conformal block!)
    - can add screenings
    - transformation s_δ maps CBs of exponential fields (with screening insertions) to CBs with degenerate external dimensions

Riccati/Chazy solutions (continued)

More general situation: for

$$\boldsymbol{\theta} = \frac{1}{2}\left(\eta, N, N-z-z', z'-z+\eta\right), \qquad N \in \mathbb{Z}_{>0},$$
$$\boldsymbol{\sigma} = \frac{1}{2}\left(N+\eta, z+z', z'-z+N\right) \mod \mathbb{Z}$$

PVI admits a 4-parameter family of solutions [Forrester, Witte, '02]

$$\tau(t) = t^{\frac{N\eta}{2}}\,(1-t)^{\frac{N(z+z'-N)}{2}}\,\det\left[f_{j-k}\right]_{j,k=1,\ldots,N}$$

$$f_\ell = \frac{\Gamma(1-z')}{\Gamma(1-\ell+\eta)\,\Gamma(1+\ell-\eta-z')}\,{}_2F_1\left(z, -\ell+\eta+z', 1-\ell+\eta, t\right) + $$
$$+ \frac{\xi\,\Gamma(1-z)}{\Gamma(1+\ell-\eta)\,\Gamma(1-\ell+\eta-z)}\,t^{\ell-\eta}\,{}_2F_1\left(z', \ell-\eta+z, 1+\ell-\eta, t\right).$$

- dimension $\Delta_t = \frac{N^2}{4}$ is degenerate (level $N+1$)
- $\xi = 0 \Rightarrow$ single conformal block $\mathcal{B}\left(\frac{\eta}{2}, \frac{N}{2}, \frac{N-z-z'}{2}, \frac{z'-z+\eta}{2}, \frac{N+\eta}{2}, t\right)$
- subsequently $\eta \to 0 \Rightarrow$ Toeplitz determinant $D_N^{(z,z')}$ for z-measures

Riccati/Chazy solutions (continued)

How to recover Gessel from AGT?

- We have $\sigma_{0t} = \theta_0 + \theta_t$. But $\mathcal{B}_{\lambda,\mu}(\boldsymbol{\theta},\sigma)$ contains the product

$$\prod_{(i,j)\in\mu} (\theta_0 + \theta_t - \sigma + i - j)$$

It vanishes for any non-empty μ as it contains $(i,j) = (1,1)$.

- remaining AGT sum over λ simplifies to

$$\sum_{\lambda\in\mathbb{Y}} t^{|\lambda|} \prod_{(i,j)\in\lambda} \frac{i-j+N}{i-j+N+\eta} \frac{(i-j+z)(i-j+z'+\eta)}{h_\lambda^2(i,j)},$$

and can be restricted to λ with $\lambda_1 \leq N$ thanks to $\theta_t - \theta_0 + \sigma_{0t} = N$

- finally letting $\eta \to 0$ we get

$$D_N^{(z,z')} = \sum_{\lambda\in\mathbb{Y},\lambda_1\leq N} t^{|\lambda|} \prod_{(i,j)\in\lambda} \frac{(i-j+z)(i-j+z')}{h_\lambda^2(i,j)}$$

Transformation $t \leftrightarrow 1-t$:

- $\xi = 0$:

1 CB at $t=0$ (internal dimension $\sigma = \frac{N}{2}$) $\rightarrow$ $N+1$ CBs at $t=1$ (internal dimensions $\sigma_k = \theta_1 - \frac{N}{2} + k,\ k = 0,\ldots,N$)

- $\xi \neq 0$: $N+1$ CBs at $t=0$ (internal dimensions $\sigma_k = \theta_0 + \frac{N}{2} - k,\ k = 0,\ldots,N$)

Picard solutions

- $\omega_{0t} = \omega_{1t} = \omega_{01} = \omega_4 = 0$
- parameters can be Backlund transformed to $\boldsymbol{\theta}_{\text{Picard}} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$
- dimensions $\Delta_\nu = \theta_\nu^2$ correspond to Ashkin-Teller conformal block [Zamolodchikov, '86]

$$\mathcal{B}\left(\boldsymbol{\theta}_{\text{Picard}}, \sigma; t\right) = \frac{\left(16q\right)^{\sigma^2}}{t^{\frac{1}{8}}\left(1-t\right)^{\frac{1}{8}}\vartheta_3(0|\tau)}$$

where $q = e^{i\pi\tau}$, $\tau = \dfrac{iK'(t)}{K(t)}$ and

$$K(t) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-tx^2)}}, \qquad K'(t) = K(1-t).$$

- structure constants C_n and parameter s_{0t} simplify to

$$C_n \sim 2^{-4(\sigma_{0t}+n)^2}(-s_{0t})^n, \qquad s = -e^{\pm 2\pi i \sigma_{1t}}$$

- conformal expansion $\tau(t) = \sum_{n\in\mathbb{Z}} C_n \mathcal{B}(\boldsymbol{\theta}, \sigma_{0t}+n, t)$ then gives theta function series so that finally

$$\tau_{\text{Picard}}(t) = \text{const}\cdot\frac{q^{\sigma_{0t}^2}}{t^{\frac{1}{8}}(1-t)^{\frac{1}{8}}}\frac{\vartheta_3\left(\sigma_{0t}\pi\tau \pm \sigma_{1t}\pi|\tau\right)}{\vartheta_3(0|\tau)}.$$

(this indeed coincides with Picard tau function [Kitaev, Korotkin, '98])

Conclusions

1. Isomonodromic tau functions are Fourier transforms of $c = 1$ conformal blocks and their irregular analogs
2. Nekrasov-AGT combinatorial formulas provide series representations for general solutions of Painlevé VI, V, III and an efficient tool of their numerical computation

More questions

1. connection problem for Painlevé tau functions/fusion matrix of $c = 1$ conformal blocks ✓
2. increase rank and genus
3. irregular braiding/fusion transformations
4. ...