Spectra of 2D Sigma Models on Symmetric Spaces

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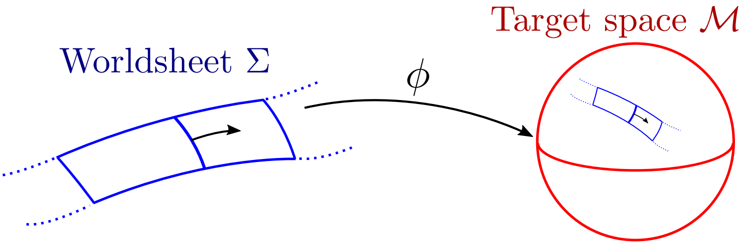
CFT and Integrability
In Memory of Alexei Zamolodchikov

Based on arXiv:1211.2238 and arXiv:1308.5981
with C. Candu and V. Schomerus

Sigma Models on Symmetric Spaces

Preliminaries



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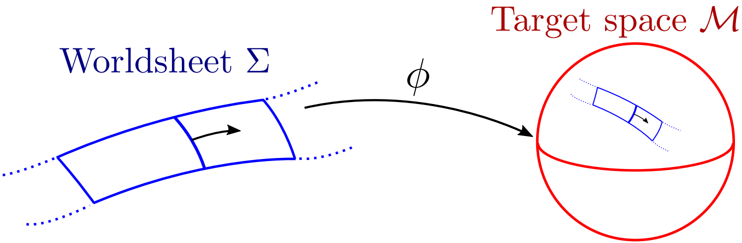

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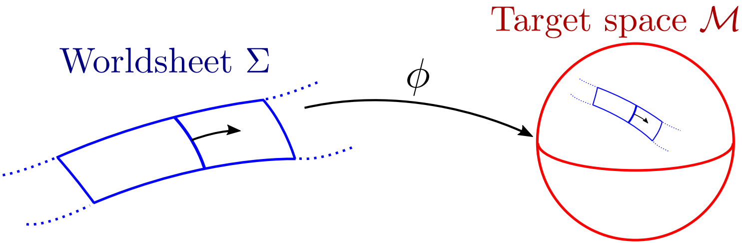

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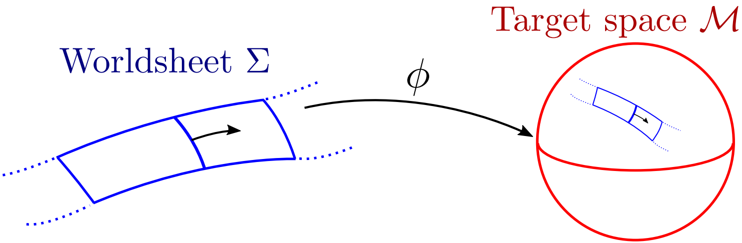

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$$\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{m} \qquad [\mathfrak{h}\,,\,\mathfrak{h}] \subset \mathfrak{h} \qquad [\mathfrak{h}\,,\,\mathfrak{m}] \subset \mathfrak{m} \qquad [\mathfrak{m}\,,\,\mathfrak{m}] \subset \mathfrak{h}$$

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- All loop conformal models (without $\mathcal{N} = 1$ supersymmetry)

$$\mathrm{PSU}(n|n) \qquad \mathrm{OSP}(2n+2|2n) \qquad D(2,1;\alpha)$$

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- Classically integrable

Reformulation

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Maurer-Cartan form

$$\omega = g^{-1}dg \Longrightarrow \text{Pullback: } \phi^\star \imath^\star \omega = Jdz + \bar{J}d\bar{z}$$

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Projectors:

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Gauge transformations: $g(z,\bar{z}) \mapsto g(z,\bar{z})h(z,\bar{z})$

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Action

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Space of States

Free theory limit

In the $R \to \infty$ limit, the sigma model is free

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In the free theory limit, they have zero energy
Example: $S^1 = SO(2)/SO(1)$,

$$V_\alpha =: e^{i\alpha\phi} : \qquad h_\alpha = \bar{h}_\alpha = \frac{\alpha^2}{8R^2} \xrightarrow{R\to\infty} 0$$

Excited states can be obtained by acting with derivatives

Covariant formulation

Covariant formulation for the tachyonic states

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$$\Longrightarrow L_2(S^2) \cong \bigoplus_{j=0}^{\infty} (j) \qquad \text{zero energy states for } R \to \infty$$

Reformulation of the states

Generalize to sections on G/H with fibre W_λ

$$\Gamma_\lambda = \left\{ f \in L_2(G) : f(gh) = R_\lambda(h^{-1}) f(g) \; \forall h \in H \right\} \cong \bigoplus_\Lambda n_{\Lambda\lambda} V_\Lambda \otimes W_\lambda$$

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combine with currents

$$\jmath_{\mathbf{n}} = \partial_A^{n_1} \jmath \otimes \cdots \partial_A^{n_r} \jmath \in \mathfrak{m} \otimes \cdots \otimes \mathfrak{m}, \qquad \mathbf{n} = \{n_1, \ldots, n_r\}$$

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and get the states

$$\Phi_{(\Lambda,\lambda,\mu,\bar{\mu})} = \mathsf{c}_{\lambda\mu\bar{\mu}} \left[f_{\Lambda\lambda}(\imath(\phi)) \otimes \mathsf{p}_\mu(\jmath_{\mathbf{n}}) \otimes \mathsf{p}_{\bar{\mu}}(\bar{\jmath}_{\bar{\mathbf{n}}}) \right]$$

$$\mathsf{p}_\mu : \mathfrak{m} \otimes \cdots \otimes \mathfrak{m} \to W_\mu \qquad \mathsf{c}_{\lambda\mu\bar{\mu}} : W_\lambda \otimes W_\mu \otimes W_{\bar{\mu}} \to \mathbb{C}$$

Space of states

The states

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- In that limit $\Phi_{\vec{\Lambda}}$ has the left/right conformal dimension

$$h = (n_1 + 1) + \cdots + (n_r + 1) \qquad \bar{h} = (\bar{n}_1 + 1) + \cdots + (\bar{n}_{\bar{r}} + 1)$$

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- No winding modes!
  Free boson: $h_{m,w} = \frac{1}{2}\left(\frac{m}{2R} + wR\right)^2 \Rightarrow$ they decouple

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The currents $\jmath$ and $\bar{\jmath}$ become abelian

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H compact

$$\mathcal{H}_{G/H} = \left[L_2(G) \otimes \mathcal{A} \otimes \bar{\mathcal{A}}\right]^{H-\text{invariants}}$$

where $\mathcal{A}$ is the Fock space generated by the abelian $\jmath$

Partition functions

The free partition function

$$\mathcal{Z}^{\text{free}}_{G/H}(q,\bar{q}\,|\,\mathbf{x}) = \text{str}_{\mathcal{H}}\Big(\prod_i x_i^{J_i} q^{L_0}\bar{q}^{\bar{L}_0}\Big)$$
$$= \Big(\,\mathcal{Z}_{L_2(G)}(\mathbf{x},\mathbf{y})\mathcal{Z}_{\mathcal{J}}(q\,|\,\mathbf{y})\mathcal{Z}_{\bar{\mathcal{J}}}(\bar{q}\,|\,\mathbf{y})\,\Big)^{H-\text{invariants}}$$

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Sections (G compact)

$$\mathcal{Z}_{L_2(G)}(\mathbf{x},\mathbf{y}) = \sum_{\Lambda} \underbrace{\chi_{S_\Lambda}(\mathbf{x})}_{\text{irred.}}\ \underbrace{\chi_{P_\Lambda}(\mathbf{y})}_{\text{proj. covers}}$$

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Currents

$$\mathcal{Z}_{\jmath}(q\,|\,\mathbf{y}) = \prod_{n=1}^{\infty} \frac{1}{\text{sdet}[1 - R_{\mathfrak{m}}(\mathbf{y})q^n]} = \sum_{\mu} B_\mu(q)\chi_{W_\mu}(\mathbf{y})$$

Partition functions

The complete partition function

$$\mathcal{Z}^{\text{free}}_{G/H}(q,\bar{q}\,|\,\mathbf{x}) = \sum_{\Lambda,\lambda,\mu,\bar{\mu}} \chi_{S_\Lambda}(\mathbf{x}) n_{\Lambda\lambda} \mathcal{N}_{\lambda\mu\bar{\mu}} B_\mu(q) B_{\bar{\mu}}(\bar{q})$$

$$N_{\lambda\mu\bar{\mu}} = \dim \mathrm{Hom}(W_\lambda \otimes W_\mu \otimes W_{\bar{\mu}}, \mathbb{C}) \qquad P_\Lambda = \sum_\lambda n_{\Lambda\lambda} W_\lambda$$

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Application to $S^{M|N}$ - independent checks

$$\mathcal{Z}^{\text{free}}_{S^{N-1}}(q,\bar{q}\,|\,\mathbf{x}) = \sum_{\substack{\Lambda,\lambda,l \\ \mu,\bar{\mu},\alpha,\beta,\gamma}} sd_\Lambda(\mathbf{x}) c^{\Lambda}_{\lambda(l)} c^{\lambda}_{\alpha\beta} c^{\mu}_{\beta\gamma} c^{\bar{\mu}}_{\alpha\gamma} \frac{s_\mu(\mathbf{q})}{\prod_{i<j}(1-q^{i+j})} \frac{s_{\bar{\mu}}(\bar{\mathbf{q}})}{\prod_{i<j}(1-\bar{q}^{i+j})}$$

$$= \left(\lim_{u\to 1} \frac{1-u^2}{\det(1-\mathbf{x}u)}\right) \times \prod_{n=1}^{\infty} \frac{|1-q^n|^2}{|\det(1-\mathbf{x}q^n)|^2}$$

1-loop partition function

Claim: One-loop partition function

$$\mathcal{Z}_{G/H}^{\text{1-loop}}(q,\bar{q}\,|\,\mathbf{x}) = \sum_{\Lambda,\lambda,\mu,\bar{\mu}} \chi_{S_\Lambda}(\mathbf{x}) n_{\Lambda\lambda} \mathcal{N}_{\lambda\mu\bar{\mu}} B_\mu(q) B_{\bar{\mu}}(\bar{q}) (q\bar{q})^{\delta h_{\Lambda\lambda\mu\bar{\mu}}}$$

Perturbation theory at 1-loop

Background field expansion

Background field

$g = g_0 e^{i\phi}$ for $\phi \in \mathfrak{m}$

$$g^{-1}dg = \underbrace{id\phi}_{\in\mathfrak{m}} + \frac{1}{2}\underbrace{[\phi\,,\,d\phi]}_{\in\mathfrak{h}} - \frac{i}{6}\underbrace{[\phi\,,\,[\phi\,,\,d\phi]]}_{\in\mathfrak{m}} + \cdots$$

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Action at 1-loop

$$\mathcal{S} = \frac{R^2}{2}\int\frac{d^2z}{\pi}\,(\jmath,\bar{\jmath}) = \frac{R^2}{2}\int\frac{d^2z}{\pi}\Big[\underbrace{(\partial\phi,\bar{\partial}\phi)}_{\text{free}} + \underbrace{\frac{1}{3}\left([\phi\,,\,\partial\phi]\,,\,[\phi\,,\,\bar{\partial}\phi]\right)}_{=\Omega-\text{interaction}} + \cdots\Big]$$

$$\langle\,\phi(x,\bar{x})\otimes\phi(y,\bar{y})\,\rangle \;=\; -\frac{t_i\otimes t^i}{R^2}\,\log\frac{|x-y|^2}{\epsilon^2}, \qquad \mathfrak{m} = \mathrm{span}(t_i)_{i=1}^{\dim\mathfrak{m}}$$

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Currents at 1-loop

$$\jmath(x) = i\partial\phi(x) + \cdots$$

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States

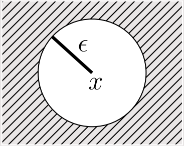
$$\Phi_{\vec{\Lambda}}(x|g_0) = \mathsf{c}_{\lambda\mu\bar{\mu}}\Big[f_{\Lambda\lambda}(g_0) + iL_{\Lambda}(\mathrm{Ad}_{g_0}\phi)f_{\Lambda\lambda}(g_0)\Big] \otimes \mathsf{p}_{\mu}(\jmath_{\mathbf{n}}) \otimes \mathsf{p}_{\bar{\mu}}(\bar{\jmath}_{\bar{\mathbf{n}}}) + \cdots$$

Correlation functions

2-point function

$$\left\langle \Phi_{\vec{\Lambda}}(x) \otimes \Phi_{\vec{\Lambda}'}(y) \right\rangle = \int_{G/H} d\boldsymbol{\mu}(g_0 H) \left\langle \Phi_{\vec{\Lambda}}(x \,|\, g_0) \otimes \Phi_{\vec{\Lambda}'}(y \,|\, g_0) e^{-\mathcal{S}_{\text{int}}} \right\rangle$$

One loop anomalous dimensions



$$\left\langle \Phi_{\vec{\Lambda}}(x) \otimes \Phi_{\vec{\Lambda}'}(y) \right\rangle_1 = 2 \left\langle \delta\mathbf{h} \cdot \Phi_{\vec{\Lambda}}(x) \otimes \Phi_{\vec{\Lambda}'}(y) \right\rangle_0 \log \left| \frac{\epsilon}{x-y} \right|^2 + \text{non-log.}$$

One loop perturbation theory

$$\left\langle\, \Phi_{\vec{\Lambda}}(x) \otimes \Phi_{\vec{\Lambda}'}(y)\, \right\rangle_0 = \int_{G/H} d\boldsymbol{\mu}(g_0 H)\, (\mathsf{d}_{\lambda\mu\bar{\mu}} \otimes \mathsf{d}_{\lambda'\mu'\bar{\mu}'}) I_0$$

$$\mathsf{d}_{\lambda\mu\bar{\mu}} = \mathsf{c}_{\lambda\mu\bar{\mu}}\big(\, \mathbb{1} \otimes \mathsf{p}_{\mu} \otimes\, \mathsf{p}_{\bar{\mu}}\big)$$

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$$I_0 = \langle f_{\Lambda\lambda}(g_0) \otimes \jmath_{\mathbf{m}}(x) \otimes \bar{\jmath}_{\bar{\mathbf{m}}}(\bar{x}) \otimes f_{\Lambda'\lambda'}(g_0) \otimes \jmath_{\mathbf{n}}(y) \otimes \bar{\jmath}_{\bar{\mathbf{n}}}(\bar{y}) \, \rangle$$

One loop perturbation theory

$$\left\langle\, \Phi_{\vec{\Lambda}}(x) \otimes \Phi_{\vec{\Lambda}'}(y) \,\right\rangle_0 = \int_{G/H} d\boldsymbol{\mu}(g_0 H)\, (\mathsf{d}_{\lambda\mu\bar{\mu}} \otimes \mathsf{d}_{\lambda'\mu'\bar{\mu}'}) I_0$$

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$$\left\langle\, \Phi_{\vec{\Lambda}}(x) \otimes \Phi_{\vec{\Lambda}'}(y) \,\right\rangle_1 = \int_{G/H} d\boldsymbol{\mu}(g_0 H)\, (\mathsf{d}_{\lambda\mu\bar{\mu}} \otimes \mathsf{d}_{\lambda'\mu'\bar{\mu}'}) \left(I_1' + I_1''\right)$$

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$$I_1' = -\Big\langle L_\Lambda(\mathrm{Ad}_{g_0}\phi(x,\bar{x})) f_{\Lambda\lambda} \otimes \jmath_{\mathbf{m}}(x) \otimes \bar{\jmath}_{\bar{\mathbf{m}}}(\bar{x}) \otimes \otimes L_{\Lambda'}(\mathrm{Ad}_{g_0}\phi(y,\bar{y})) f_{\Lambda'\lambda'} \otimes \jmath_{\mathbf{n}}(y) \otimes \bar{\jmath}_{\bar{\mathbf{n}}}(\bar{y}) \Big\rangle$$

$$I_1'' = -\int_{\mathbb{C}_\epsilon} \frac{d^2 z}{\pi} \, \langle f_{\Lambda\lambda} \otimes \jmath_{\mathbf{m}}(x) \otimes \bar{\jmath}_{\bar{\mathbf{m}}}(\bar{x}) \otimes f_{\Lambda'\lambda'} \otimes \jmath_{\mathbf{n}}(y) \otimes \bar{\jmath}_{\bar{\mathbf{n}}}(\bar{y}) \, \Omega(z,\bar{z}) \rangle$$

Anomalous dimensions

Final result

$$\delta \mathbf{h}_{\Lambda\lambda\mu\bar{\mu}} = \frac{1}{2R^2} \underbrace{\left(\mathrm{Cas}_{\mathfrak{g}}^{\Lambda} - \mathrm{Cas}_{\mathfrak{h}}^{\lambda}\right)}_{I_1' - \text{Bochner Laplacian}} + \frac{1}{2R^2} \underbrace{\left(\mathrm{Cas}_{\mathfrak{h}}^{\lambda} - \mathrm{Cas}_{\mathfrak{h}}^{\mu} - \mathrm{Cas}_{\mathfrak{h}}^{\bar{\mu}}\right)}_{I_1''}$$

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Anomalous dimensions

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$$\delta \mathbf{h}_{\Lambda\lambda\mu\bar{\mu}r\bar{r}} = \frac{1}{2R^2}\underbrace{\left(\mathrm{Cas}^{\Lambda}_{\mathfrak{g}} - \mathrm{Cas}^{\lambda}_{\mathfrak{h}}\right)}_{I_1' - \text{Bochner Laplacian}} + \frac{1}{2R^2}\underbrace{\left(\mathrm{Cas}^{\lambda}_{\mathfrak{h}} - \mathrm{Cas}^{\mu}_{\mathfrak{h}} - \mathrm{Cas}^{\bar{\mu}}_{\mathfrak{h}}\right)}_{I_1''}$$

$$= \frac{1}{2R^2}\left(\mathrm{Cas}^{\Lambda}_{\mathfrak{g}} - \mathrm{Cas}^{\mu}_{\mathfrak{h}} - \mathrm{Cas}^{\bar{\mu}}_{\mathfrak{h}}\right) + (r+\bar{r})\,\mathrm{Cas}^{\mathfrak{m}}_{\mathfrak{h}} \qquad \text{non-conformal}$$

$$\Phi_{(\Lambda,\lambda,\mu,\bar{\mu},r,\bar{r})} = c_{\lambda\mu\bar{\mu}}\left[f_{\Lambda\lambda}(\imath(\phi)) \otimes p_{\mu}(j_{\mathbf{n}}) \otimes p_{\bar{\mu}}(\bar{\jmath}_{\bar{\mathbf{n}}})\right], \quad |\mathbf{n}| = r,\ |\bar{\mathbf{n}}| = \bar{r}$$

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Generalization

Same results with $\mathcal{N} = 1$ worldsheet supersymmetry.

Conclusion and Outlook

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Results

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- Comparizon with WZW model results
  $S^{2N-1|2N}$ sigma model dual to the $\mathrm{OSP}(2N+2|2N)$ Gross-Neveu model?

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- Higher-loops

Thank you