$U_q(\widehat{\mathfrak{gl}}_N)$ Nested Bethe Vectors

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see also arXiv:1206.4931, 1207.0956, 1210.0768, 1211.3968, 1304.7602, 1311.3500, 1312.1488

Motivations

- Main problem – calculation of the correlation functions and/or calculation of the form-factors of the local operators
- In the models where algebraic Bethe ansatz is applicable this problem can be solved by
  - knowledge of the Bethe vectors and the dual ones
  - effective formulas for the scalar products of these vectors
- For $U_q(\widehat{\mathfrak{gl}}_2)$ related integrable models the possibility of form-factor calculations is provided by
  - simple form of the Bethe vectors in terms of the monodromy matrix elements
  - existence of the effective (determinant) Slavnov's formula for the scalar product for these Bethe vectors
- For $U_q(\widehat{\mathfrak{gl}}_N)$ related integrable models situation is much more complicated

$U_q(\widehat{\mathfrak{gl}}_N)$ related integrable models

- $N \times N$ monodromy matrix $T_{ij}(z)$
- $\mathrm{R}(u,v;q)\cdot(T(u)\otimes\mathbf{1})\cdot(\mathbf{1}\otimes T(v)) = (\mathbf{1}\otimes T(v))\cdot(T(u)\otimes\mathbf{1})\cdot\mathrm{R}(u,v;q)$
- $\mathrm{R}(u,v;q) = \mathrm{f}_q(u,v)\sum_{1\le i\le N}\mathrm{E}_{ii}\otimes\mathrm{E}_{ii} + \sum_{1\le i<j\le N}(\mathrm{E}_{ii}\otimes\mathrm{E}_{jj}+\mathrm{E}_{jj}\otimes\mathrm{E}_{ii})$

$$+\sum_{1\le i<j\le N}\left(\mathrm{g}^{(l)}_q(u,v)\mathrm{E}_{ij}\otimes\mathrm{E}_{ji}+\mathrm{g}^{(r)}_q(u,v)\mathrm{E}_{ji}\otimes\mathrm{E}_{ij}\right)$$

- $\mathrm{f}_q(u,v) = \dfrac{qu-q^{-1}v}{u-v}, \quad \mathrm{g}_q(u,v) = \dfrac{(q-q^{-1})}{u-v}$

$$\mathrm{g}^{(l)}_q(u,v) = u\mathrm{g}_q(u,v)\,, \quad \mathrm{g}^{(r)}_q(u,v) = v\mathrm{g}_q(u,v)$$

$U_q(\widehat{\mathfrak{gl}}_N)$ related integrable models (continued)

- The off-shell Bethe vectors (BV) belong to the quantum space V of the integrable models. They are polynomials on the monodromy matrix elements T_{ij} depending on the $N-1$ sets of the Bethe parameters $\bar{n} = \{n_1, \dots, n_{N-1}\}$:

$$\begin{aligned}\bar{t}_{\bar{n}} &= \left\{t_1^1, \dots, t_{n_1}^1; t_1^2, \dots, t_{n_2}^2; \dots\dots; t_1^{N-1}, \dots, t_{n_{N-1}}^{N-1}\right\} \\ &= \left\{\bar{t}^1; \bar{t}^2; \dots; \bar{t}^{N-1}\right\}\end{aligned}$$

- V is identified with a representation space of the algebra of the monodromy matrix elements generated by the vector $|0\rangle$ satisfying

$$T_{j,i}(z)|0\rangle = 0, \quad j > i, \quad T_{i,i}(z)|0\rangle = \lambda_i(z)|0\rangle, \quad i = 1, \dots, N$$

- Dual (or left) off-shell BV belong to the dual space V^*. It is generated by the matrix elements $T_{i,j}(z)$ acting on a vector $\langle 0|$ satisfying

$$\langle 0|T_{i,j}(z) = 0, \quad j > i, \quad \langle 0|T_{i,i}(z) = \lambda_i(z)\langle 0|, \quad i = 1, \dots, N$$

What is known for $U_q(\widehat{\mathfrak{gl}}_N)$ case?

- Nested (hierarchical) Bethe ansatz
  - P.Kulish, N.Reshetikhin *J.Phys. A: Math. Gen.* **16** (1983) L591
  - P.Kulish, N.Reshetikhin *J. Sov. Math.* **34**:5 (1982) 1948
- Tarasov-Varcheko construction of the Bethe vectors
  - V.Tarasov, A.Varchenko *Algebra and Analysis* **2** (1995) no.2, 275
- Hierarchical relations for the Bethe vectors on the level of the evaluation modules
  - V.Tarasov, A.Varchenko *SIGMA* **9** (2013) 048, math/0702277
- Isomorphism between *RLL* and current formulations of QAA $U_q(\widehat{\mathfrak{gl}}_N)$
  - V.Drinveld *Soviet Math. Dokl.* **36** (1988), 212
  - J.Ding, I.B.Frenkel *Comm. Math. Phys.* **156** (1993), 277
- Expressions for $U_q(\widehat{\mathfrak{gl}}_N)$-invariant Bethe vectors
  - S.Khoroshkin, S.Pakuliak, *J. of Math of Kyoto University* **48** n.2 (2008) 277
  - A.Os'kin, S.Pakuliak, A.Silantyev *Algebra and Analysis* **21** n.4 (2009) 196

T-operator formulation of $U_q(\widehat{\mathfrak{gl}}_N)$

Reshetikhin, N., Semenov-Tian-Shansky, M. *Lett. Math. Phys.* **19** (1990) 133

T-operators

$$T^{\pm}(z) = \sum_{k=0}^{\infty} \sum_{i,j=1}^{N} \mathrm{E}_{ij} \otimes T_{ij}^{\pm}[\pm k] z^{\mp k}$$

$$T_{ji}^{+}[0] = T_{ij}^{-}[0] = 0, \quad 1 \leq i < j \leq N, \quad T_{kk}^{+}[0]\, T_{kk}^{-}[0] = 1, \quad 1 \leq k \leq N$$

Commutation relations $(c = 0,\ \mu, \nu = \pm)$

$$\mathrm{R}(u,v) \cdot (T^{\mu}(u) \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes T^{\nu}(v)) = (\mathbf{1} \otimes T^{\nu}(v)) \cdot (T^{\mu}(u) \otimes \mathbf{1}) \cdot \mathrm{R}(u,v)$$

Borel subalgebras and coalgebraic structure

$$T^{\pm}[n] \in U_q(\mathfrak{b}^{\pm}) \subset U_q(\widehat{\mathfrak{gl}}_N)\,, \quad \Delta\left(T_{ij}^{\pm}(u)\right) = \sum_{k=1}^{N} T_{kj}^{\pm}(u) \otimes T_{ik}^{\pm}(u)$$

Two Gauss decompositions of $U_q(\widehat{\mathfrak{gl}}_N)$ monodromy matrix

$$T^{\pm}_{a,b}(t) = \mathrm{F}^{\pm}_{b,a}(t)k^{+}_{b}(t) + \sum_{b<m\le N} \mathrm{F}^{\pm}_{m,a}(t)k^{+}_{m}(t)\mathrm{E}^{\pm}_{b,m}(t) \quad a<b$$

$$T^{\pm}_{b,b}(t) = k^{\pm}_{b}(t) + \sum_{b<m\le N} \mathrm{F}^{\pm}_{m,b}(t)k^{\pm}_{m}(t)\mathrm{E}^{\pm}_{b,m}(t)$$

$$T^{\pm}_{a,b}(t) = k^{\pm}_{a}(t)\mathrm{E}^{\pm}_{b,a}(t) + \sum_{a<m\le N} \mathrm{F}^{\pm}_{m,a}(t)k^{\pm}_{m}(t)\mathrm{E}^{\pm}_{b,m}(t) \quad a>b$$

$$T^{\pm}_{a,b}(t) = \hat{\mathrm{F}}^{\pm}_{b,a}(t)\hat{k}^{+}_{a}(t) + \sum_{1\le m<a} \hat{\mathrm{F}}^{\pm}_{b,m}(t)\hat{k}^{+}_{m}(t)\hat{\mathrm{E}}^{\pm}_{m,a}(t) \quad a<b$$

$$T^{\pm}_{a,a}(t) = \hat{k}^{\pm}_{a}(t) + \sum_{1\le m<a} \hat{\mathrm{F}}^{\pm}_{a,m}(t)\hat{k}^{\pm}_{m}(t)\hat{\mathrm{E}}^{\pm}_{m,a}(t)$$

$$T^{\pm}_{a,b}(t) = \hat{k}^{\pm}_{b}(t)\hat{\mathrm{E}}^{\pm}_{b,a}(t) + \sum_{1\le m<b} \hat{\mathrm{F}}^{\pm}_{b,m}(t)\hat{k}^{\pm}_{m}(t)\hat{\mathrm{E}}^{\pm}_{m,i}(t) \quad a>b$$

$$U_q(\widehat{\mathfrak{gl}}_{N-1}) \hookrightarrow U_q(\widehat{\mathfrak{gl}}_N)$$

Current realization of $U_q(\widehat{\mathfrak{gl}}_N)$

$$F_i(t) = \mathrm{F}^+_{i+1,i}(t) - \mathrm{F}^-_{i+1,i}(t) \qquad E_i(t) = \mathrm{E}^+_{i,i+1}(t) - \mathrm{E}^-_{i,i+1}(t)$$

Commutation relations

$$\ldots \quad \ldots$$

$$(qz - q^{-1}w)F_i(z)F_i(w) = F_i(w)F_i(z)(q^{-1}z - qw)$$

$$(q^{-1}z - qw)F_i(z)F_{i+1}(w) = F_{i+1}(w)F_i(z)(z - w)$$

$$k_i^{\pm}(z)F_i(w)\left(k_i^{\pm}(z)\right)^{-1} = \frac{q^{-1}z - qw}{z - w}F_i(w)$$

$$[E_i(z), F_j(w)] = (q - q^{-1})\delta_{i,j}\delta(z/w)(k_i^+(z)/k_{i+1}^+(z) - k_i^-(w)/k_{i+1}^-(w))$$

$$\ldots \quad \ldots$$

Current Hopf structure

$$\Delta^{(D)}(F_i(z)) = F_i(z) \otimes 1 + k_{i+1}^+(z)\left(k_i^+\right)^{-1} \otimes F_i(z)$$

$$\ldots \quad \ldots$$

Second current realization of $U_q(\widehat{\mathfrak{gl}}_N)$

$$\hat{F}_i(t) = \hat{\mathrm{F}}^+_{i+1,i}(t) - \hat{\mathrm{F}}^-_{i+1,i}(t) \qquad \hat{E}_i(t) = \hat{\mathrm{E}}^+_{i,i+1}(t) - \hat{\mathrm{E}}^-_{i,i+1}(t)$$

Commutation relations

$$\dots \quad \dots$$

$$(q^{-1}z - qw)\hat{F}_i(z)\hat{F}_i(w) = \hat{F}_i(w)\hat{F}_i(z)(qz - q^{-1}w)$$

$$(z - w)\hat{F}_i(z)\hat{F}_{i+1}(w) = \hat{F}_{i+1}(w)\hat{F}_i(z)(q^{-1}z - qw)$$

$$\hat{k}^\pm_i(z)\hat{F}_i(w)\left(\hat{k}^\pm_i(z)\right)^{-1} = \frac{q^{-1}z - qw}{z - w}\hat{F}_i(w)$$

$$[\hat{E}_i(z), \hat{F}_j(w)] = (q - q^{-1})\delta_{i,j}\delta(z/w)(\hat{k}^-_{i+1}(z)/\hat{k}^-_i(z) - \hat{k}^+_{i+1}(w)/\hat{k}^+_i(w))$$

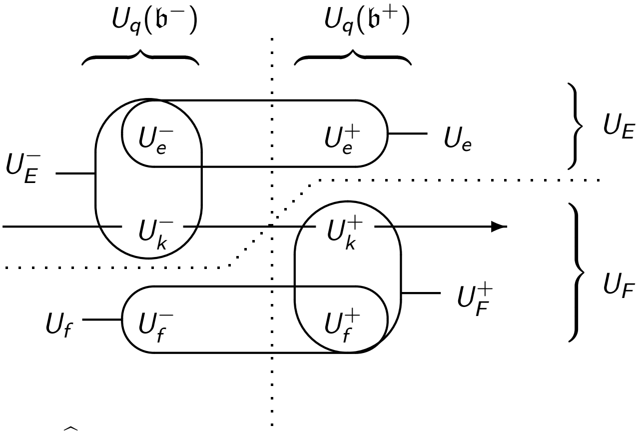
$$\dots \quad \dots$$

Second current Hopf structure

$$\hat{\Delta}^{(D)}(\hat{F}_i(z)) = 1 \otimes \hat{F}_i(z) + \hat{F}_i(z) \otimes \hat{k}^+_i(z)\left(\hat{k}^+_{i+1}\right)^{-1}$$

$$\dots \quad \dots$$

Different type Borel subalgebras



Subalgebras of $U_q(\widehat{\mathfrak{gl}}_N)$. The vertical dotted line separates the standard Borel subalgebras. The horizontal dotted line separates the current Borel subalgebras. The horizontal solid axis indicates the increasing of the current generators modes. Ovals denote different subalgebras in the $U_q(\widehat{\mathfrak{gl}}_N)$ standard and current Borel subalgebras.

Notations

$$T_{i,j}(\bar{t}) = \prod_{t_k \in \bar{t}} T_{i,j}(t_k), \qquad \lambda_i(\bar{t}) = \prod_{t_k \in \bar{t}} \lambda_i(t_k)$$

$$\mathsf{f}(\bar{t}, \bar{t}') = \prod_{t_j \in \bar{t}} \prod_{t_k \in \bar{t}'} \mathsf{f}(t_j, t_k)$$

$$\beta(\bar{t}_{\bar{n}}) = \prod_{k=1}^{N-1} \prod_{1 \le \ell < \ell' \le n_k} \mathsf{f}(t^k_{\ell'}, t^k_{\ell}), \qquad \gamma(\bar{t}_{\bar{n}}) = \frac{\beta(\bar{t}_{\bar{n}})}{\prod_{i=1}^{N-2} \mathsf{f}(\bar{t}^{i+1}, \bar{t}^i)}$$

$$\mathsf{K}_n(\bar{x}|\bar{y}) = \frac{\prod_{1 \le i,j \le k}(qx_i - q^{-1}y_j)}{\prod_{1 \le i < j \le k}(x_i - x_j)(y_j - y_i)} \cdot \det\left[\frac{q - q^{-1}}{(x_i - y_j)(qx_i - q^{-1}y_j)}\right]$$

$$\mathsf{K}^{(l)}_n(\bar{x}|\bar{y}) = \prod_{i=1}^{n} x_i \cdot \mathsf{K}_n(\bar{x}|\bar{y}), \qquad \mathsf{K}^{(r)}_n(\bar{x}|\bar{y}) = \prod_{i=1}^{n} y_i \cdot \mathsf{K}_n(\bar{x}|\bar{y})$$

Tarasov-Varchenko construction of the off-shell BV

Tarasov, V., Varchenko, A. *Algebra and Analysis* **2** (1995) no.2, 275

$$\mathbb{T}(\bar{u}) = T^{(1)}(u_1)\cdots T^{(M)}(u_M)\cdot \overleftarrow{\prod_{1\le i<j\le M}} \mathrm{R}^{(ji)}(u_j,u_i) \in \left(\mathrm{End}(\mathbb{C}^N)\right)^{\otimes M}\otimes U_q(\mathfrak{b}^+)$$

Off-Shell BV

$$\mathbb{B}^{\bar{n}}(\bar{t}_{\bar{n}}) = \mathrm{tr}_{\mathbb{C}^{\otimes M}}\left(\mathbb{T}(t^1_1,\ldots,t^1_{n_1};\ \ldots\ ;t^{N-1}_1,\ldots,t^{N-1}_{n_{N-1}})\ \mathrm{E}_{21}^{\otimes n_1}\otimes\cdots\otimes \mathrm{E}_{N,N-1}^{\otimes n_{N-1}}\right)|0\rangle$$

$$\bar{u}\to\bar{t}_{\bar{n}} = \left\{t^1_1,\ldots,t^1_{n_1};t^2_1,\ldots,t^2_{n_2};\ldots\ldots;t^{N-1}_1,\ldots,t^{N-1}_{n_{N-1}}\right\},\quad M=\sum_k n_k$$

$$T^+_{ij}(t)|0\rangle = 0,\quad i>j,\quad T^+_{kk}(t)v = \lambda_k(t)|0\rangle$$

Bethe vectors in terms of the current generators

- B.Enriquez, V.Rubtsov. *Israel J. Math* **112** (1999) 61
- B.Enriquez, S.Khoroshkin, S.Pakuliak. *Comm. Math. Phys.* **276** (2007) 691

$$\mathbb{B}^{\bar n}(\bar t_{\bar n}) = \gamma(\bar t_{\bar n}) P_f^+ \left(F_{N-1}(t^{N-1}_{n_{N-1}}) \cdots F_{N-1}(t^{N-1}_1) \cdots F_1(t^1_{n_1}) \cdots F_1(t^1_1) \right) \prod_{i=1}^{N-1} k_{i+1}(\bar t^i)|0\rangle$$

$$\mathbb{C}^{\bar n}(\bar t_{\bar n}) = \gamma(\bar t_{\bar n}) \langle 0| \prod_{i=1}^{N-1} k_{i+1}(\bar t^i) P_e^+ \left(E_1(t^1_1) \cdots E_1(t^1_{n_1}) \cdots E_{N-1}(t^{N-1}_1) \cdots E_{N-1}(t^{N-1}_{n_{N-1}}) \right)$$

$$\mathbb{B}^{\bar n}(\bar t_{\bar n}) = \gamma(\bar t_{\bar n}) \hat P_f^+ \left(\hat F_1(t^1_1) \cdots \hat F_1(t^1_{n_1}) \cdots \hat F_{N-1}(t^{N-1}_1) \cdots \hat F_{N-1}(t^{N-1}_{n_{N-1}}) \right) \prod_{i=1}^{N-1} \hat k_i(\bar t^i)|0\rangle$$

$$\mathbb{C}^{\bar n}(\bar t_{\bar n}) = \gamma(\bar t_{\bar n}) \langle 0| \prod_{i=1}^{N-1} \hat k_i(\bar t^i)\, \hat P_e^+ \left(\hat E_{N-1}(t^{N-1}_{n_{N-1}}) \cdots \hat E_{N-1}(t^{N-1}_1) \cdots \hat E_1(t^1_{n_1}) \cdots \hat E_1(t^1_1) \right)$$

$$\Delta\left(P_f^+(\mathcal{F}) \right) |0\rangle \otimes |0\rangle = (P_f^+ \otimes P_f^+) \Delta^{(D)}(\mathcal{F}) |0\rangle \otimes |0\rangle$$

Combinatorial formulas for BV [TV,KP,OPS]

$$[\bar m]=\{m^i_j\}\ \text{for}\ 1\le i\le j\le N-1$$

$$m^i_i\ge m^i_{i+1}\ge\cdots\ge m^i_{N-1}\ge m^i_N=0\,,\quad n_i=\sum_{j=1}^{i}m^j_i\,,\quad i=1,\dots,N-1$$

$$\bar{\mathbf{m}}^j=\{n_1,n_2,\dots,n_j,m^1_{j+1}+\cdots+m^j_{j+1},\dots,m^1_{N-1}+\cdots+m^j_{N-1}\}$$

$$\mathbb{B}^{\bar n}(\bar t_{\bar n})=\sum_{[\bar m]}\mathrm{Sym}_{\ \bar t_{\bar n}}\Bigg(\beta(\bar t_{\bar n})\prod_{1\le j\le i<N}\Bigg([(m^j_i-m^j_{i+1})!]^{-1}\prod_{n_j-m^j_i<\ell'<\ell\le n_j-m^j_{i+1}}\mathsf{f}(t^j_\ell,t^j_{\ell'})^{-1}\Bigg)$$

$$\times\prod_{i=2}^{N-1}\Bigg(\prod_{j=1}^{i-1}\Bigg(\prod_{\ell=0}^{m^j_i-1}\mathsf{g}^{(l)}(t^i_{\mathbf{m}^j_i-\ell},t^{i-1}_{\mathbf{m}^j_{i-1}-\ell})\prod_{\ell'=\mathbf{m}^j_{i-1}-\ell+1}^{n_{i-1}}\mathsf{f}(t^i_{\mathbf{m}^j_i-\ell},t^{i-1}_{\ell'})\Bigg)\Bigg)$$

$$\times\overrightarrow{\prod_{1\le j\le N-1}}\Bigg(\overleftarrow{\prod_{N-1\ge i\ge j}}\Bigg(\prod_{\ell=n_j-m^j_i+1}^{n_j-m^j_{i+1}}T_{j,i+1}(t^j_\ell)\Bigg)\Bigg)\prod_{j=1}^{N-1}\prod_{\ell=1}^{n_j-m^j_j}\lambda_j(t^j_\ell)\Bigg)|0\rangle$$

$$\mathrm{Sym}_{\ \bar t_{\bar n}}\ G(\bar t_{\bar n})=\sum_{\sigma\in S_{\bar n}}G({}^{\sigma}\bar t_{\bar n})\,,\quad \sigma=\{\sigma^1,\sigma^2,\dots,\sigma^{N-1}\}$$

$$[\bar{s}] = \{s_i^j\} \text{ for } 1 \le i \le j \le N-1$$

$$0 = s_0^i \le s_1^i \le \cdots \le s_{i-1}^{i-1} \le s_i^i\,, \quad n_i = \sum_{j=i}^{N-1} s_i^j\,, \quad i = 1,\ldots,N-1$$

$$\bar{\mathsf{s}}^j = \bar{s}^j + \bar{s}^{j+1} + \cdots + \bar{s}^{N-2} + \bar{s}^{N-1}\,, \quad j = 1,\ldots,N-1$$

$$\mathbb{B}^{\bar{n}}(\bar{t}_{\bar{n}}) = \sum_{[\bar{s}]} \mathrm{Sym}_{\ \bar{t}_{\bar{n}}} \Bigg(\beta(\bar{t}_{\bar{n}}) \prod_{1\le i\le j<N} \Bigg([(s_i^j - s_{i-1}^j)!]^{-1} \prod_{s_{i-1}^j<\ell'<\ell\le s_i^j} \mathsf{f}(t_\ell^j, t_{\ell'}^j)^{-1} \Bigg)$$

$$\times \prod_{j=2}^{N-1} \Bigg(\prod_{i=1}^{j-1} \Bigg(\prod_{\ell=1}^{s_i^j} \mathsf{g}^{(r)}(t^{i+1}_{n_{i+1}-\mathsf{s}^j_{i+1}+\ell}, t^i_{n_i-\mathsf{s}^j_i+\ell}) \prod_{\ell'=1}^{n_{i+1}-\mathsf{s}^j_{i+1}+\ell-1} \mathsf{f}(t^{i+1}_{\ell'}, t^i_{n_i-\mathsf{s}^j_i+\ell}) \Bigg)\Bigg)$$

$$\times \overleftarrow{\prod_{N-1\ge j\ge 1}} \Bigg(\overrightarrow{\prod_{1\le i\le j}} \Bigg(\prod_{\ell=s_{i-1}^j+1}^{s_i^j} T_{i,j+1}(t_\ell^j) \Bigg)\Bigg) \prod_{j=1}^{N-1} \prod_{\ell=s_j^j}^{n_j} \lambda_{j+1}(t_\ell^j) \Bigg) |0\rangle$$

From permutations to partitions

For $m^j_i=\delta^j_i n_i$ we have

$$\frac{1}{n_1!n_2!\cdots n_{N-1}!}\ \mathrm{Sym}_{\ \bar t_{\bar n}}\Big(T_{1,2}(\bar t^1)\,T_{2,3}(\bar t^2)\cdots T_{N-1,N}(\bar t^{N-1})\Big)$$

$$1\le i\le k\le j\le N-1\,,\quad \bar t^k=\bigcup_{i=1}^{k}\ \bigcup_{j=k}^{N-1}\ \bar t^k_{i,j}$$

$$\#\bar t^k_{i,j}=m^i_j-m^i_{j+1}\ \text{ for }\ i=1,\ldots,k\ \text{ and }\ j=k,\ldots,N-1\,,\quad \forall k$$

$$\#\bar t^k_{i,j}=s^j_i-s^j_{i-1}\ \text{ for }\ i=1,\ldots,k\ \text{ and }\ j=k,\ldots,N-1\,,\quad \forall k$$

$$i,j\prec i',j'\quad\text{if}\quad i<i',\ \ \forall j,j'\quad\text{or if}\quad i=i',j<j'$$

$$i,j\prec^t i',j'\quad\text{if}\quad j<j',\ \ \forall i,i'\quad\text{or if}\quad j=j',i<i'$$

BV as sums over partitions of the Bethe parameters

$$\mathcal{B}^{\bar n}(\bar t_{\bar n}) = \sum_{\rm part} \prod_{k=1}^{N-1} \prod_{i,j\prec i',j'} \mathsf{f}(\bar t^k_{i',j'}, \bar t^k_{i,j}) \prod_{k=2}^{N-1} \left(\prod_{i,j\prec i',j'} \mathsf{f}(\bar t^k_{i,j}, \bar t^{k-1}_{i',j'}) \prod_{i<j} \mathsf{K}^{(l)}(\bar t^k_{i,j}|\bar t^{k-1}_{i,j}) \right)$$
$$\times \prod^{\longrightarrow}_{1\le k\le N-1} \left(\prod^{\longleftarrow}_{N\ge j>k} T_{k,j}(\bar t^k_{k,j-1}) \right) \prod_{k=2}^{N-1} \prod_{i,j\prec k,k} T_{k,k}(\bar t^k_{i,j})$$

$$\widehat{\mathcal{B}}^{\bar n}(\bar t_{\bar n}) = \sum_{\rm part} \prod_{k=1}^{N-1} \prod_{i,j\prec^t i',j'} \mathsf{f}(\bar t^k_{i',j'}, \bar t^k_{i,j}) \prod_{k=2}^{N-1} \left(\prod_{i,j\prec^t i',j'} \mathsf{f}(\bar t^k_{i,j}, \bar t^{k-1}_{i',j'}) \prod_{i<j} \mathsf{K}^{(r)}(\bar t^k_{i,j}|\bar t^{k-1}_{i,j}) \right)$$
$$\times \prod^{\longleftarrow}_{N-1\ge k\ge 1} \left(\prod^{\longrightarrow}_{1\le j\le k} T_{j,k+1}(\bar t^k_{j,k}) \right) \prod_{k=1}^{N-2} \prod_{k,k\prec^t i,j} T_{k+1,k+1}(\bar t^k_{i,j})$$

$$\mathbb{B}^{\bar n}(\bar t_{\bar n}) = \mathcal{B}^{\bar n}(\bar t_{\bar n})|0\rangle = \widehat{\mathcal{B}}^{\bar n}(\bar t_{\bar n})|0\rangle$$

Examples of the partitions for $U_q(\widehat{\mathfrak{gl}}_5)$ and 'nice' identity

$$\begin{aligned}
\bar{t}^1 &= \bar{t}^1_{1,1} \cup \bar{t}^1_{1,2} \cup \bar{t}^1_{1,3} \cup \bar{t}^1_{1,4} \\
\bar{t}^2 &= \bar{t}^2_{1,2} \cup \bar{t}^2_{1,3} \cup \bar{t}^2_{1,4} \cup \bar{t}^2_{2,2} \cup \bar{t}^2_{2,3} \cup \bar{t}^2_{2,4} \\
\bar{t}^3 &= \bar{t}^3_{1,3} \cup \bar{t}^3_{1,4} \cup \bar{t}^3_{2,3} \cup \bar{t}^3_{2,4} \cup \bar{t}^3_{3,3} \cup \bar{t}^3_{3,4} \\
\bar{t}^4 &= \bar{t}^4_{1,4} \cup \bar{t}^4_{2,4} \cup \bar{t}^4_{3,4} \cup \bar{t}^4_{4,4}
\end{aligned}$$

$$\begin{aligned}
\bar{t}^4 &= \bar{t}^4_{4,4} \cup \bar{t}^4_{3,4} \cup \bar{t}^4_{2,4} \cup \bar{t}^4_{1,4} \\
\bar{t}^3 &= \bar{t}^3_{3,4} \cup \bar{t}^3_{2,4} \cup \bar{t}^3_{1,4} \cup \bar{t}^3_{3,3} \cup \bar{t}^3_{2,3} \cup \bar{t}^3_{2,4} \\
\bar{t}^2 &= \bar{t}^2_{2,4} \cup \bar{t}^2_{1,4} \cup \bar{t}^3_{2,3} \cup \bar{t}^3_{1,3} \cup \bar{t}^3_{2,2} \cup \bar{t}^2_{1,2} \\
\bar{t}^1 &= \bar{t}^1_{1,4} \cup \bar{t}^1_{1,3} \cup \bar{t}^1_{1,2} \cup \bar{t}^1_{1,1}
\end{aligned}$$

$$\mathrm{Sym}_{\bar{y}} \left(\prod_{\ell<\ell'} \mathsf{f}(y_{\ell'}, y_\ell) \prod_{\ell=1}^{n} \mathsf{g}(y_\ell, x_\ell) \prod_{\ell<\ell'} \mathsf{f}(y_\ell, x_{\ell'}) \right) = \mathsf{K}_n(\bar{y}|\bar{x})$$

Properties of R-matrix

$$
\mathrm{g}^{(l)}_{q^{-1}}(v,u)=\mathrm{g}^{(r)}_{q}(u,v),\qquad \mathrm{g}^{(l)}_{q^{-1}}(u^{-1},v^{-1})=\mathrm{g}^{(r)}_{q}(u,v)
$$

$$
\mathrm{f}_{q^{-1}}(v,u)=\mathrm{f}_{q}(u,v),\qquad \mathrm{f}_{q^{-1}}(u^{-1},v^{-1})=\mathrm{f}_{q}(u,v)
$$

$$
\mathrm{K}^{(l)}_{q^{-1}}(\bar v|\bar u)=\mathrm{K}^{(r)}_{q}(\bar u|\bar v),\qquad \mathrm{K}^{(l)}_{q^{-1}}(\bar u^{-1}|\bar v^{-1})=\mathrm{K}^{(r)}_{q}(\bar u|\bar v)
$$

$$
\mathrm{R}_{12}(u,v)\,\mathrm{R}_{21}(v,u)=\mathrm{f}_q(u,v)\mathrm{f}_q(v,u)\mathbf{1}\otimes\mathbf{1}
$$

$$
U_1\,U_2\,\mathrm{R}_{12}(u,v)\,U_1^{-1}\,U_2^{-1}\;=\;\mathrm{R}_{21}(u,v),\qquad U=\sum_{i=1}^{N}\mathrm{E}_{i,N+1-i}
$$

$$
\mathrm{R}_{21}(v,u;q)=\mathrm{R}_{21}(u^{-1},v^{-1};q)=\mathrm{R}_{12}(u,v;q^{-1})
$$

$$
\mathrm{R}_{12}(u,v)^{t_1t_2}\;=\;\mathrm{R}_{21}(u,v)
$$

$$
\mathrm{R}_{12}(v^{-1},u^{-1})=\mathrm{R}_{12}(u,v)
$$

$U_q(\widehat{\mathfrak{gl}}_N)$ algebra morphisms

- The map φ defined by

$$\varphi\big(T(u)\big) \;=\; U\,\tilde{T}^t(u)\,U^{-1}$$

defines an isomorphism from $U_q(\widehat{\mathfrak{gl}}_N)$ to $U_{q^{-1}}(\widehat{\mathfrak{gl}}_N)$.

- The map ψ given by

$$\psi\big(T(u)\big) \;=\; \tilde{T}^t(u^{-1})$$

defines an anti-isomorphism from $U_q(\widehat{\mathfrak{gl}}_N)$ to $U_{q^{-1}}(\widehat{\mathfrak{gl}}_N)$

Here $T(u) \in U_q(\widehat{\mathfrak{gl}}_N)$ and $\tilde{T}(u) \in U_{q^{-1}}(\widehat{\mathfrak{gl}}_N)$

Dual (left) pre-BV

$$\mathcal{C}^{\bar n}(\bar t_{\bar n}) = \sum_{\rm part} \prod_{k=1}^{N-1} \prod_{i,j\prec i',j'} \mathrm{f}(\bar t^k_{i',j'},\bar t^k_{i,j}) \prod_{k=2}^{N-1} \left(\prod_{i,j\prec i',j'} \mathrm{f}(\bar t^k_{i,j},\bar t^{k-1}_{i',j'}) \prod_{i<j} \mathrm{K}^{(r)}(\bar t^k_{i,j}|\bar t^{k-1}_{i,j}) \right)$$
$$\times \prod_{k=2}^{N-1} \prod_{i,j\prec k,k} T_{k,k}(\bar t^k_{i,j}) \overleftarrow{\prod_{N-1\geq k\geq 1}} \left(\overrightarrow{\prod_{k<j\leq N}} T_{j,k}(\bar t^k_{k,j-1}) \right),$$

$$\widehat{\mathcal{C}}^{\bar n}(\bar t_{\bar n}) = \sum_{\rm part} \prod_{k=1}^{N-1} \prod_{i,j\prec^t i',j'} \mathrm{f}(\bar t^k_{i',j'},\bar t^k_{i,j}) \prod_{k=2}^{N-1} \left(\prod_{i,j\prec^t i',j'} \mathrm{f}(\bar t^k_{i,j},\bar t^{k-1}_{i',j'}) \prod_{i<j} \mathrm{K}^{(l)}(\bar t^k_{i,j}|\bar t^{k-1}_{i,j}) \right)$$
$$\times \prod_{k=1}^{N-2} \prod_{k,k\prec^t i,j} T_{k+1,k+1}(\bar t^k_{i,j}) \overrightarrow{\prod_{1\leq k\leq N-1}} \left(\overleftarrow{\prod_{1\leq j\leq k}} T_{k+1,j}(\bar t^k_{j,k}) \right).$$

$$\mathbb{C}^{\bar n}(\bar t_{\bar n}) = \langle 0|\mathcal{C}^{\bar n}(\bar t_{\bar n}) = \langle 0|\widehat{\mathcal{C}}^{\bar n}(\bar t_{\bar n})$$

Main Proposition of arXiv:1310.3253

- The morphism φ relates the universal off-shell pre-BV

$$\varphi\Big(\widehat{\mathcal{B}}^{\bar n}_q(\bar t_{\bar n})\Big) = \mathcal{B}^{\,^\omega\bar n}_{q^{-1}}(^\omega\bar t_{\bar n})$$

$$\omega : \bar t_{\bar n} \to\ ^\omega\bar t_{\bar n} = \Big\{t^{N-1}_1, \dots, t^{N-1}_{n_{N-1}}; \dots; t^2_1, \dots, t^2_{n_2}; t^1_1, \dots, t^1_{n_1}\Big\}$$

$$\omega : \bar n \to\ ^\omega\bar n = \{n_{N-1}, n_{N-2}, \dots, n_2, n_1\}$$

- Dual pre-BV $\mathcal{C}^{\bar n}(\bar t_{\bar n})$ and $\widehat{\mathcal{C}}^{\bar n}(\bar t_{\bar n})$ are related to the pre-BV $\mathcal{B}^{\bar n}(\bar t_{\bar n})$ and $\widehat{\mathcal{B}}^{\bar n}(\bar t_{\bar n})$ by the antimorphism ψ

$$\psi\Big(\mathcal{B}^{\bar n}_q(\bar t_{\bar n})\Big) = \mathcal{C}^{\bar n}_{q^{-1}}(\bar t^{-1}_{\bar n}) \quad \text{and} \quad \psi\Big(\widehat{\mathcal{B}}^{\bar n}_q(\bar t_{\bar n})\Big) = \widehat{\mathcal{C}}^{\bar n}_{q^{-1}}(\bar t^{-1}_{\bar n}),$$

$$\bar t^{-1}_{\bar n} = \left\{\left(t^1_1\right)^{-1}, \dots, \left(t^1_{n_1}\right)^{-1}; \dots; \left(t^{N-1}_1\right)^{-1}, \dots, \left(t^{N-1}_{n_{N-1}}\right)^{-1}\right\}$$

$$\bar t^1 = \bar t^1_{1,1} \cup \bar t^1_{1,2} \to \bar u_{\rm II} \cup \bar u_{\rm I} = \bar u$$
$$\bar t^2 = \bar t^2_{1,2} \cup \bar t^2_{2,2} \to \bar v_{\rm I} \cup \bar v_{\rm II} = \bar v$$
$$\#\bar u_{\rm I} = \#\bar v_{\rm I} \qquad n_1 = a \qquad n_2 = b$$

$$\mathbb{B}^{a,b}(\bar u,\bar v) = \sum_{\rm part} {\rm K}^{(l)}(\bar v_{\rm I}|\bar u_{\rm I}){\rm f}(\bar u_{\rm I},\bar u_{\rm II}){\rm f}(\bar v_{\rm II},\bar v_{\rm I})\lambda_2(\bar v_{\rm I})\,T_{13}(\bar u_{\rm I})\,T_{12}(\bar u_{\rm II})\,T_{23}(\bar v_{\rm II})|0\rangle$$
$$\mathbb{B}^{a,b}(\bar u,\bar v) = \sum_{\rm part} {\rm K}^{(r)}(\bar v_{\rm I}|\bar u_{\rm I}){\rm f}(\bar u_{\rm I},\bar u_{\rm II}){\rm f}(\bar v_{\rm II},\bar v_{\rm I})\lambda_2(\bar u_{\rm I})\,T_{13}(\bar v_{\rm I})\,T_{23}(\bar v_{\rm II})\,T_{12}(\bar u_{\rm II})|0\rangle$$

$$\mathbb{C}^{a,b}(\bar u,\bar v) = \sum_{\rm part} {\rm K}^{(r)}(\bar v_{\rm I}|\bar u_{\rm I}){\rm f}(\bar u_{\rm I},\bar u_{\rm II}){\rm f}(\bar v_{\rm II},\bar v_{\rm I})\lambda_2(\bar v_{\rm I})\langle 0|\,T_{32}(\bar v_{\rm II})\,T_{21}(\bar u_{\rm II})\,T_{31}(\bar u_{\rm I})$$
$$\mathbb{C}^{a,b}(\bar u,\bar v) = \sum_{\rm part} {\rm K}^{(l)}(\bar v_{\rm I}|\bar u_{\rm I}){\rm f}(\bar u_{\rm I},\bar u_{\rm II}){\rm f}(\bar v_{\rm II},\bar v_{\rm I})\lambda_2(\bar u_{\rm I})\langle 0|\,T_{21}(\bar u_{\rm II})\,T_{32}(\bar v_{\rm II})\,T_{31}(\bar v_{\rm I})$$

Multiple monodromy matrix elements action onto BV

$$\mathbb{B}^{a,b}(\bar u,\bar v)\to\frac{\mathbb{B}^{a,b}(\bar u,\bar v)}{\mathsf{f}(\bar v,\bar u)\lambda_2(\bar u)\lambda_2(\bar v)}$$

$$\{\bar v,\bar w\}=\bar\xi,\ \{\bar u,\bar w\}=\bar\eta \text{ and } \#\bar w=n$$

$$T_{12}(\bar w)\mathbb{B}^{a,b}(\bar u;\bar v)=\lambda_2(\bar w)\sum\frac{\mathsf{f}(\bar\xi_{\mathrm{II}},\bar\xi_{\mathrm{I}})}{\mathsf{f}(\bar w,\bar\xi_{\mathrm{I}})}\mathsf{K}_n^{(r)}(\bar w|\bar\xi_{\mathrm{I}})\,\mathbb{B}^{a+n,b}(\bar\eta;\bar\xi_{\mathrm{II}})$$

$$\bar\xi\Rightarrow\{\bar\xi_{\mathrm{I}},\bar\xi_{\mathrm{II}}\}\text{ with }\#\bar\xi_{\mathrm{I}}=n$$

$$T_{32}(\bar w)\mathbb{B}^{a,b}(\bar u;\bar v)=\lambda_2(\bar w)\sum\frac{\lambda_3(\bar\xi_{\mathrm{I}})}{\lambda_2(\bar\xi_{\mathrm{I}})}\,\frac{\mathsf{f}(\bar\xi_{\mathrm{I}},\bar\xi_{\mathrm{II}})\mathsf{f}(\bar\xi_{\mathrm{I}},\bar\xi_{\mathrm{III}})\mathsf{f}(\bar\xi_{\mathrm{III}},\bar\xi_{\mathrm{II}})\mathsf{f}(\bar\eta_{\mathrm{I}},\bar\eta_{\mathrm{II}})}{\mathsf{f}(\bar\xi_{\mathrm{I}},\bar\eta)\mathsf{f}(\bar\eta_{\mathrm{I}},\bar w)\mathsf{f}(\bar w,\bar\xi_{\mathrm{II}})}$$
$$\times\mathsf{K}_n^{(l)}(\bar\eta_{\mathrm{I}}|\bar w)\,\mathsf{K}_n^{(l)}(\bar\xi_{\mathrm{I}}|\bar\eta_{\mathrm{I}})\,\mathsf{K}_n^{(r)}(\bar w|\bar\xi_{\mathrm{II}})\,\mathbb{B}^{a,b-n}(\bar\eta_{\mathrm{II}};\bar\xi_{\mathrm{III}})$$

$$\bar\xi\Rightarrow\{\bar\xi_{\mathrm{I}},\bar\xi_{\mathrm{II}},\bar\xi_{\mathrm{III}}\}\text{ with }\#\bar\xi_{\mathrm{I}}=\#\bar\xi_{\mathrm{II}}=n;\quad\bar\eta\Rightarrow\{\bar\eta_{\mathrm{I}},\bar\eta_{\mathrm{II}}\}\text{ with }\#\bar\eta_{\mathrm{I}}=n$$

Reshetikhin's formula for the scalar products

$$S_{a,b}(\bar{u}^C;\bar{v}^C|\bar{u}^B;\bar{v}^B) = \mathbb{C}^{a,b}(\bar{u}^C;\bar{v}^C)\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)$$

$$\mathrm{r}_1(u) = \frac{\lambda_1(u)}{\lambda_2(u)} \quad \text{and} \quad \mathrm{r}_3(u) = \frac{\lambda_3(u)}{\lambda_2(u)}$$

$$S_{a,b}(\bar{u}^C;\bar{v}^C|\bar{u}^B;\bar{v}^B) = \sum \frac{\mathrm{r}_1(\bar{u}^C_{\mathrm{II}})\mathrm{r}_1(\bar{u}^B_{\mathrm{I}})\mathrm{r}_3(\bar{v}^C_{\mathrm{II}})\mathrm{r}_3(\bar{v}^B_{\mathrm{I}})}{\mathrm{f}(\bar{v}^C,\bar{u}^C)\mathrm{f}(\bar{v}^B,\bar{u}^B)} W_{\mathrm{part}}\begin{pmatrix} \bar{u}^C_{\mathrm{II}},\bar{u}^B_{\mathrm{II}}, & \bar{u}^C_{\mathrm{I}},\bar{u}^B_{\mathrm{I}} \\ \bar{v}^C_{\mathrm{I}},\bar{v}^B_{\mathrm{I}}, & \bar{v}^C_{\mathrm{II}},\bar{v}^B_{\mathrm{II}} \end{pmatrix}$$

$$W_{\mathrm{part}}\begin{pmatrix} \bar{u}^C_{\mathrm{II}},\bar{u}^B_{\mathrm{II}},\bar{u}^C_{\mathrm{I}},\bar{u}^B_{\mathrm{I}} \\ \bar{v}^C_{\mathrm{I}},\bar{v}^B_{\mathrm{I}},\bar{v}^C_{\mathrm{II}},\bar{v}^B_{\mathrm{II}} \end{pmatrix} = \mathrm{f}(\bar{u}^B_{\mathrm{II}},\bar{u}^B_{\mathrm{I}})\mathrm{f}(\bar{u}^C_{\mathrm{I}},\bar{u}^C_{\mathrm{II}})\mathrm{f}(\bar{v}^B_{\mathrm{I}},\bar{v}^B_{\mathrm{II}})\mathrm{f}(\bar{v}^C_{\mathrm{II}},\bar{v}^C_{\mathrm{I}})\mathrm{f}(\bar{v}^C_{\mathrm{I}},\bar{u}^C_{\mathrm{I}})\mathrm{f}(\bar{v}^B_{\mathrm{II}},\bar{u}^B_{\mathrm{II}})$$
$$\times\, \mathrm{Z}^{(l)}_{a-k,n}(\bar{u}^C_{\mathrm{II}};\bar{u}^B_{\mathrm{II}}|\bar{v}^C_{\mathrm{I}};\bar{v}^B_{\mathrm{I}})\; \mathrm{Z}^{(r)}_{k,b-n}(\bar{u}^B_{\mathrm{I}};\bar{u}^C_{\mathrm{I}}|\bar{v}^B_{\mathrm{II}};\bar{v}^C_{\mathrm{II}})$$

$$\mathrm{Z}^{(l)}_{a,b}(\bar{t};\bar{x}|\bar{s};\bar{y}) = (-q)^{-b}\sum \mathrm{K}^{(r)}_b(\bar{s}|\bar{w}_{\mathrm{I}}q^2)\mathrm{K}^{(l)}_a(\bar{w}_{\mathrm{II}}|\bar{t})\mathrm{K}^{(l)}_b(\bar{y}|\bar{w}_{\mathrm{I}})\mathrm{f}(\bar{w}_{\mathrm{I}},\bar{w}_{\mathrm{II}})$$
$$\bar{w} = \{\bar{s},\bar{x}\} = \{\bar{w}_{\mathrm{I}},\bar{w}_{\mathrm{II}}\} \qquad \#\bar{w}_{\mathrm{I}} = b \quad \#\bar{w}_{\mathrm{II}} = a$$

Identities

Bethe equations

$$\bar{u} \to \{\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}) \qquad \bar{v} \to \{\bar{v}_{\mathrm{I}}, \bar{v}_{\mathrm{II}})$$

$$\mathrm{r}_1(\bar{u}_{\mathrm{I}}) = \frac{\mathrm{f}(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}})}{\mathrm{f}(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{I}})}\ \mathrm{f}(\bar{v}, \bar{u}_{\mathrm{I}}) \qquad \mathrm{r}_3(\bar{v}_{\mathrm{I}}) = \frac{\mathrm{f}(\bar{v}_{\mathrm{II}}, \bar{v}_{\mathrm{I}})}{\mathrm{f}(\bar{v}_{\mathrm{I}}, \bar{v}_{\mathrm{II}})}\ \mathrm{f}(\bar{v}_{\mathrm{I}}, \bar{u})$$

Main Lemma

$$\bar{\gamma},\ \bar{\alpha},\ \bar{\beta},\ \#\alpha = m_1,\ \#\beta = m_2,\ \#\gamma = m_1 + m_2$$

$$\sum \mathrm{K}_{m_1}(\bar{\gamma}_{\mathrm{I}}|\bar{\alpha})\mathrm{K}_{m_2}(\bar{\beta}|\bar{\gamma}_{\mathrm{II}})\mathrm{f}(\bar{\gamma}_{\mathrm{II}}, \bar{\gamma}_{\mathrm{I}}) = (-q)^{-m_1}\mathrm{f}(\bar{\gamma}, \bar{\alpha})\mathrm{K}_{m_1+m_2}(\{\bar{\alpha}q^{-2}, \bar{\beta}\}|\bar{\gamma})$$

$$\bar{\gamma} \Rightarrow \{\bar{\gamma}_{\mathrm{I}}, \bar{\gamma}_{\mathrm{II}}\},\ \#\bar{\gamma}_{\mathrm{I}} = m_1,\ \#\bar{\gamma}_{\mathrm{II}} = m_2$$

Conclusion

- 'Good' formulas for the off-shell Bethe vectors in terms of sums over partitions are found in case of the quantum integrable models associated with $U_q(\widehat{\mathfrak{gl}}_N)$ trigonometric R-matrix
- These formulas will hopefully help to address the problem of calculation of the scalar products of the nested off-shell Bethe vectors
- This can help to find determinant formulas for the form-factors of the local operators in the integrable systems with $U_q(\widehat{\mathfrak{gl}}_N)$ symmetries (see arXiv:1211.3968 and arXiv:1312.1488 for the rational case)

Thank you for your attention