

On Macdonald polynomials in form factors approach

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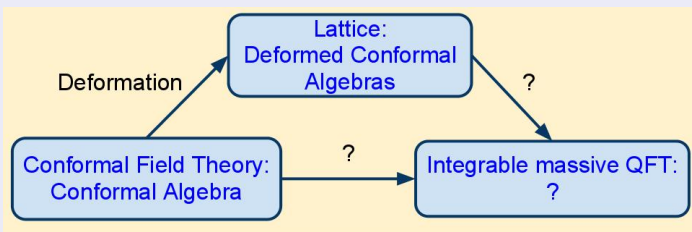
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Based on work: M. Lashkevich, YP, JHEP 1309 (2013) 095

Al. Zamolodchikov conference, CQUEST,
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Dynamical Symmetry in 2D integrable models

Space of fields



Number of local operators

- CFT: Character of Virasoro algebra $Tr q^{L_0}$
- Lattice: Trace of corner transfer matrix $Tr q^{L_0}$
- Massive FT: $\sum q^{\#spin-s}$ solutions form factor equations

Massive Integrable 2D QFT

Perturbed two dimensional CFT ($c = 1 - \frac{6}{\xi(\xi+1)}$, $\Delta_\Phi = \frac{\xi-1}{(\xi+1)}$)

$$\mathcal{A} = \mathcal{A}_{CFT} + \lambda \int d^2y \Phi(y)$$

$$\langle O_1(x) O_2(0) \rangle_{massive} = \langle O_1(x) O_2(0) e^{\lambda \int d^2y \Phi(y)} \rangle_{conformal}$$

S-matrix description

$$S(\beta) = \frac{\sinh \beta + i \sin \pi \xi}{\sinh \beta - i \sin \pi \xi}$$

$$\langle O_1(x) O_2(0) \rangle_{massive} = \sum \int d\beta \cdots \langle O_1(x) | \beta, \dots \rangle \langle \beta, \dots | O_2(0) \rangle$$

Exact form factors

Axioms ($E = m \cosh \beta$, $P = m \sinh \beta$)

$$\langle O | \beta_1, \beta_2, \dots \rangle = S(\beta_1 - \beta_2) \langle O | \beta_2, \beta_1, \dots \rangle$$

$$\langle O | \beta_1 + A, \beta_2 + A, \dots \rangle = e^{sA} \langle O | \beta_1, \beta_2, \dots \rangle$$

$$\langle O | \beta_1 + 2\pi i, \beta_2, \dots, \beta_n \rangle = \langle O | \beta_2, \dots, \beta_n, \beta_1 \rangle$$

$$\text{Res}_{\beta_1 = \beta_2 + i\pi} \langle O | \beta_1, \beta_2, \beta_3 \dots \rangle = i(1 - \prod S(\beta_2 - \beta_j)) \langle O | \beta_3, \dots \rangle$$

...

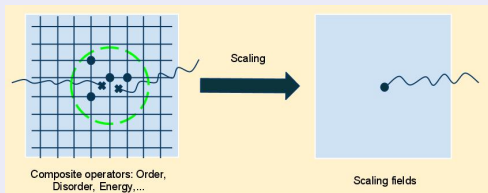
Exact answers:

- Primaries (exponential fields) studied well: sine-Gordon, sinh-Gordon, Toda, Z_N , etc.
- Descendants: many questions

Questions on descendants

Composite operators

$$\sigma(x_1) \cdots \mu(x_n) \cdots = \sum O_j(x_0), \quad |x_m - x_n| \ll R_c$$



Number of fields is "the same" as for $\lambda = 0$, but ...

$$\langle O_1(x) O_2(0) \rangle_{massive} = \langle O_1(x) O_2(0) e^{\lambda \int d^2y \Phi(y)} \rangle_{conformal}$$

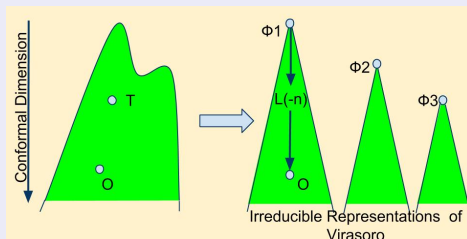
- No Virasoro algebra
- No $Vir \times \overline{Vir}$: $\bar{\partial} T \neq 0$ ($\sim \lambda \partial \Phi$)
- Sectors are mixed $[O] \sim [O] \oplus [O']$ $[\Phi]$ (resonances)

Space of States in CFT. BPZ

Virasoro Algebra $T(z) = \sum L_n z^{-n-2}$

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^2 - 1)n\delta_{n+m}$$

Primaries Φ_k and Descendants $L_{-n} \cdots \bar{L}_{-m} \cdots \Phi_k$



$$L_0|\Phi_k\rangle = \Delta_k|\Phi_k\rangle$$

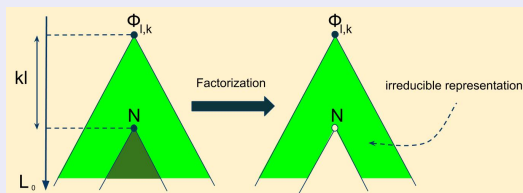
$$L_n|\Phi_k\rangle = 0 \quad (n > 0)$$

Null vectors in CFT

Kac spectrum $\Phi_{l,k}$ ($c = 1 - \frac{6}{\xi(\xi+1)}$)

$$\Delta_{l,k} = \frac{((\xi + 1)l - \xi k)^2 - 1}{4\xi(\xi + 1)}$$

Null vectors $N_{l,k} = (L_{-1}^{lk} + \dots)|\Phi_{l,k}\rangle$



$$L_0|N_{l,k}\rangle = (\Delta_{l,k} + lk)|N_{l,k}\rangle$$

$$L_n|N_{l,k}\rangle = 0, \quad n > 0$$

Oscillator realization

Oscillators and zero modes $[a_m, a_n] = m\delta_{m+n}$, $[Q, a_0] = i$

- Free field

$$\phi(z) = Q - ia_0 \log z + \sum_{m \neq 0} \frac{a_m}{im} z^{-m}$$

- Screening currents

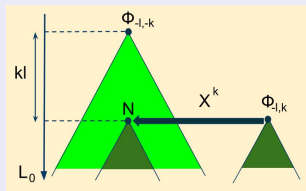
$$X = \oint \frac{dz}{2\pi iz} e^{-i\sqrt{2\frac{\xi+1}{\xi}}\phi(z)}$$

- Virasoro algebra: $[T(z), X] = 0$

$$T(z) = \frac{1}{2}\partial\phi(z)^2 + \frac{1}{\sqrt{2\xi(\xi+1)}}\partial^2\phi(z)$$

Null vectors in Fock space $\mathcal{F}_{-l,-k} = \{a_{-n_1} \cdots a_{-n_m} \Phi_{-l,-k}\}$

Singular vectors $[X^k, T(z)]|_{\mathcal{F}_{-l,k}} = 0$



- Level One $\langle \Phi_{11} | N_{11} = \langle \Phi_{11} | a_1$
- Level Two $\langle \Phi_{12} | N_{12} = \langle \Phi_{12} | (a_2 + \frac{\xi+1}{\xi} a_1^2)$
- General case $N_{l,k}$ - in terms of Jack polynomials ($a_n \rightarrow \sum_j x_j^n$)

Deformed Virasoro (CFT limit: $x \rightarrow 1$, $z \sim \text{fixed}$)

- Oscillators

$$[a_m, a_n] = \frac{1}{m} \frac{(x^{2\xi m} - x^{-2\xi m})(x^{2(\xi+1)m} - x^{-2(\xi+1)m})}{x^m + x^{-m}} \delta_{m+n}$$

- Screening currents $U(z) \sim e^{-\sum \frac{x^n + x^{-n}}{x^{\xi n} + x^{-\xi n}} a_n z^{-n}}$

$$X = \int \frac{dz}{2\pi iz} U(z)$$

- Deformed Virasoro Algebra: $[T, X] = 0$

$$T(z) = \Lambda(zx) + \Lambda^{-1}(zx^{-1}), \quad \Lambda(z) = e^{\sum a_n z^{-n}}$$

- Null vectors - Macdonald polynomials $q = x^{2\xi}, t = x^{2(\xi+1)}$

Form factors from off-critical lattice (XYZ, RSOS)

Scaling limit $x \rightarrow 1$, $z = x^{2i\beta}$

- Diagonalization of Hamiltonian $HT(x^{2i\beta}) = E(\beta)T(x^{2i\beta})$

$$E(\beta) \rightarrow M \cosh \beta$$

- ZF algebra $T(x^{2i\beta_1})T(x^{2i\beta_2}) = S(\beta_{12})T(x^{2i\beta_2})T(x^{2i\beta_1})$

$$S(\beta) \rightarrow \frac{\sinh \beta + i \sin \pi \xi}{\sinh \beta - i \sin \pi \xi}$$

- Form factors of primaries

$$Tr_{\mathcal{F}_{lk}} \left(x^{4L_0} T(x^{2i\beta_1}) \dots T(x^{2i\beta_n}) \right) \rightarrow \langle \Phi_{l,k} | \beta_1, \dots, \beta_n \rangle$$

Extra bosonic field $B = \exp \sum b_k z^{-k}$

$$\langle B(z)B(w) \rangle = \langle \Lambda(z)\Lambda(w) \rangle^{-1}$$

$$t(\beta) = T(x^{2i\beta})B(x^{2i\beta})$$

Form factors of primaries

$$\begin{aligned} \lim_{x \rightarrow -i} \langle \Phi_{l,k} | t(\beta_1) \cdots t(\beta_n) | \Phi_{l,k} \rangle &= \\ &= \lim_{x \rightarrow 1} \text{Tr}_{\mathcal{F}_{lk}} x^{4L_0} T(x^{2i\beta_1}) \cdots T(x^{2i\beta_n}) \times \text{Tr}_{\mathcal{F}_{lk}} x^{4L_0} B(x^{2i\beta_1}) \cdots B(x^{2i\beta_n}) \\ &= \langle \Phi_{l,k} | \beta_1, \dots, \beta_n \rangle / \prod_{i < j} R(\beta_i - \beta_j) \end{aligned}$$

$R(\beta)$ - two particle minimal form factor

Algebra $t(\beta) = e^{a_0} \lambda_-(\beta) + e^{-a_0} \lambda_+(\beta)$

$$\langle \lambda_+(\beta) \lambda_+(0) \rangle = \langle \lambda_-(\beta) \lambda_-(0) \rangle = 1$$

$$\langle \lambda_-(\beta) \lambda_+(0) \rangle = \langle \lambda_+(\beta) \lambda_-(0) \rangle = 1 + \frac{2i \sin \pi \xi}{e^\beta - e^{-\beta}}$$

Heisenberg Descendants a_n

- Action of oscillators $[\lambda_\pm(\beta), a_n] = (\mp)^{n+1} e^{n\beta} \lambda_\pm(\beta)$

- Left descendants ($n > 0$)

$$\langle a_{-n} \Phi_{l,k} | \beta_1, \dots, \beta_n \rangle \rightarrow \langle \Phi_{l,k} | a_n t(\beta_1) \cdots t(\beta_n) | \Phi_{l,k} \rangle$$

- Right descendants ($n > 0$)

$$\langle \bar{a}_{-n} \Phi_{l,k} | \beta_1, \dots, \beta_n \rangle \rightarrow \langle \Phi_{l,k} | t(\beta_1) \cdots t(\beta_n) a_{-n} | \Phi_{l,k} \rangle$$

- Left-right "interaction" $[a_n, \bar{a}_m] \sim \delta_{n,m}$

Example: level two null vectors

Level two descendant form factors: $N_{12} = a_1^2 + \text{const} \times a_2$

$$\langle \Phi_\alpha | a_2 | \Phi_\alpha \rangle = 0,$$

$$\langle \Phi_\alpha | a_2 t(\beta_1) | \Phi_\alpha \rangle = 2ie^{2\beta_1} \sin \pi\alpha,$$

$$\langle \Phi_\alpha | a_2 t(\beta_1) t(\beta_2) | \Phi_\alpha \rangle = 2i((e^{2\beta_1} + e^{2\beta_2}) \sin 2\pi\alpha + 2e^{\beta_1 + \beta_2} \sin \pi\xi)$$

$$\langle \Phi_\alpha | a_1^2 | \Phi_\alpha \rangle = 0,$$

$$\langle \Phi_\alpha | a_1^2 t(\beta_1) | \Phi_\alpha \rangle = 2e^{2\beta_1} \cos \pi\alpha,$$

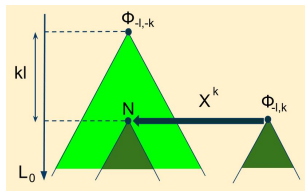
$$\langle \Phi_\alpha | a_1^2 t(\beta_1) t(\beta_2) | \Phi_\alpha \rangle = 4(e^{\beta_1} + e^{\beta_2})^2 \cos^2 \pi\alpha,$$

For $\alpha = -(\xi + 1)/2$ there is a null vector among Φ_{12} descendants:

$$\langle \Phi_{12} | N_{12} t(\beta_1) \cdots t(\beta_n) | \Phi_{12} \rangle = 0 \quad n = 0, 1, 2, \dots$$

$$\langle \Phi_{12} | N_{12} = \langle \Phi_{12} | (a_1^2 + i \tan \frac{\pi\xi}{2} a_2)$$

Structure of the space of descendants



Questions

- Arbitrary level $k \times l$ null vectors $N_{lk} = a_1^{kl} + \dots$?

$$\langle \Phi_{lk} | N_{lk} t(\beta), \dots, t(\beta_n) | \Phi_{lk} \rangle = 0$$

- Will N_{lk} create a null submodule?
- $\langle N_{lk} \bar{N}_{lk} \Phi_{lk} | \beta_1, \dots \rangle \neq 0$?

Macdonald Polynomials (q,t)

Definition

- $\lambda = \{\lambda_1, \lambda_2, \dots\}$, $\lambda_1 \geq \lambda_2 \geq \dots$, $\sum \lambda_j = n$
- $T_{q,i} f(x_1, \dots, x_i, \dots, x_N) = f(x_1, \dots, qx_i, \dots, x_N)$
- $P_\lambda(x_1, \dots, x_N)$ - symmetric polynomial of degree n

$$\sum_i \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q,i} P_\lambda = \sum t^{N-i} q^{\lambda_i} P_\lambda,$$

Examples $p_n = (1 - q^{-n}) \sum x_j^n$ ($\rightarrow a_n$ to get null vector for DVA)

$$P_{\{1\}} = p_1, \quad P_{\{2,0\}} = p_1^2 + \frac{1+q}{1-q} p_2, \quad P_{\{11\}} = p_1^2 - \frac{1+t}{1-t} p_2$$

General expression for null vectors

The proposal

Let $\lambda = \{k\}_{i=1}^l$ be $l \times k$ rectangular partition. The level $k \times l$ null vector descendant of Φ_{lk} is given by Macdonald polynomials $N_{lk} = P_\lambda|_{p_n \rightarrow a_n}$ with

$$q = e^{-i\pi(\xi+1)}, \quad t = e^{-i\pi\xi}$$

For $n = 0, 1, 2, \dots$:

$$\langle \Phi_{lk} | N_{lk} t(\beta_1) \cdots t(\beta_n) | \Phi_{lk} \rangle = 0$$

Intertwining operators

- Currents: $U(z) = U(z)^{(lattice)}|_{x=-i}$
- Charges ($x = -i$, $q = e^{-i\pi(\xi+1)}$)

$$X^{(k)} = \oint \frac{dz_1}{2\pi iz_1} \cdots \oint \frac{dz_k}{2\pi iz_k} U(z_1) \cdots U(z_k) \\ \times F(z_2/z_1) F(z_4/z_3) \cdots F(z_k/z_{k-1})$$

- $F(z) = \sum_{n=1} (-1)^{n-1} \frac{1-q^n}{1+q^n} z^n$
- $[X^{(k)}, t(z)] = 0$

Null Vectors in a single chirality

Statements on screenings charges X

- Give integral representation for Macdonald polynomials with $q = e^{-i\pi(\xi+1)}$, $t = e^{-i\pi\xi}$

$$\langle \Phi_{l,k} | N_{l,k} = \langle \Phi_{l,-k} | X^{(k)}$$

- Commute with particle creation operators $t(\beta)$ and acts as

$$X^{(k)} t(\beta) \cdots | \Phi_{l,k} \rangle = 0$$

- Create a full tower of null vectors

$$a_n X^{(k)} t(\beta) \cdots | \Phi_{l,k} \rangle = 0$$

Statements on null vectors

- Left and right chiralities are not independent
- $N_{lk} \bar{N}_{lk} \Phi_{lk}$ - resonance situation when lk is even: Al. Zamolodchikov higher equations of motion for a sinh-Gordon.

$$\langle \Phi_{l,k} | N_{l,k} t(\beta_1) \cdots t(\beta_n) \bar{N}_{l,k} | \Phi_{l,k} \rangle = C \times \langle \Phi_{l,-k} | t(\beta_1) \cdots t(\beta_n) | \Phi_{l,-k} \rangle$$

- Screening currents can be applied to get the higher conservation law equations

Toward a basis of physical fields

Higher modes of screening currents

- Definition

$$\psi_n = \oint \frac{dz}{2\pi i} z^{n-1} U(z)^{-1}$$

- Commutation relation

$$\psi_n \psi_m = -\psi_{m+2} \psi_{n-2}$$

- Intertwining property:

$$\psi_n \mathcal{X}^{(m)} = (-1)^m \mathcal{X}^{(m)} \psi_n$$

- Hypothesis on bound state equation for Lee-Yang $\xi = 2/3$

Algebraic approach to descendant form factors

- Analogues of null vectors for a breaser sector of SG
- Integral representation for the Macdonald polynomials with $q = e^{-i\pi(\xi+1)}$, $t = e^{-i\pi\xi}$
- Analysis of resonances, conservation laws, higher equation of motion