Entanglement Entropy in integrable models

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Outline

- Introduction: Von Neumann and Renyi entropies as a measure of Entanglement
- Entanglement entropy in 1+1 dim CFT
- Entanglement entropy in 1D lattice spin chains: the Corner Transfer Matrix (CTM) method
- XYZ chain exact Entanglement entropy
- Entanglement entropy in bipartite sine-Gordon
- Entanglement entropy in spin chains related to ABF and FB models
- Hamiltonians of ABF-FB chains from an algebraic point of view
- Conclusions

Entanglement: fundamental quantum property
Different reasons for interest:

1. Quantum Information $\longrightarrow$ Quantum computers
2. Quantum Phase Transitions $\longrightarrow$ universality
3. Condensed matter physics $\longrightarrow$ non-local correlators
4. Integrable models $\longrightarrow$ new playground
5. Black holes $\longrightarrow$ Information paradox & Quantum Gravity
6. **NON LOCALITY** intrinsic in Quantum Mechanics?

- **EPR paradox**: uncompleteness of QM or non-locality?
- **Bell inequalities** $\longrightarrow$ local hidden variables exist only if a certain correlation $\mathcal{P} < 2$
- Clauser Friedmann & Aspect **experiments** $\longrightarrow \mathcal{P} > 2 \Longrightarrow$ possible non-locality of QM

Quantum systems and sub-systems

- Quantum system with unique pure ground state $|0\rangle$ composed of two subsystems, A and B.
- If a state has wavefunction

$$|\psi\rangle_{A\otimes B} = |\chi\rangle_A \otimes |\phi\rangle_B$$

Separable $\Longrightarrow$ No entanglement
(i.e. measurements on B do not affect A state)

- If instead

$$|\psi\rangle_{A\otimes B} = \sum_{j=1}^{d} \lambda_j |\chi_j\rangle_A \otimes |\phi_j\rangle_B$$

with $d > 1 \Longrightarrow$ **Entangled** (measurements of B do affect A state)

Entanglement and density matrix

- Define **reduced density matrix** for subsystem A

$$\rho_A = \mathrm{Tr}_B \rho$$

Quantum entropy (Von Neumann) of Entanglement (E-Entropy)

$$S_A = -\mathrm{Tr}_A(\rho_A \log \rho_A) = S_B$$

[Bennett, Bernstein, Popescu, Schumacher 1996]

For a separable state $S_A = 0$, for a maximally entangled state it is maximal $\Longrightarrow$ S_A is a measure of Entanglement

- Area law [Srednicki 1993]

$$S_A \propto \mathrm{Area}(\partial A)$$

Renyi Entropy

Rényi Entropy

$$S_\alpha = \frac{1}{1-\alpha} \log \mathrm{Tr}_A \rho_A^\alpha$$

Introduced by hungarian mathematician Rényi in probability theory

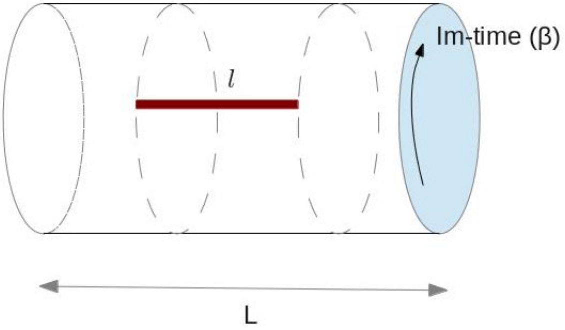
- It reduces to Von Neumann for $\alpha \to 1$
- Contains higher momenta and for $\alpha \to \infty$ the spectrum of the reduced density matrix ρ_A can be read
- link with replica trick à la Calabrese Cardy

Entanglement entorpy in CFT

[Holzhey, Larsen, Wilczek 1994 - Calabrese, Cardy 2004]

$$S(\ell) \simeq \frac{c}{3} \log \frac{\ell}{a} + U$$

c = central charge of CFT, U = non-universal constant,
a = UV-cutoff



Entanglement entopy in CFT: generalizations

Finite temperature $T = \beta^{-1}$, infinite size

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\frac{\pi\ell}{\beta}\right) + U \to \begin{cases} \ell \gg \beta & \frac{\pi c}{3}\frac{\ell}{\beta} \\ \ell \ll \beta & \frac{c}{3}\log\frac{\ell}{a} \end{cases}$$

Finite size L

$$S_A = \frac{c}{3}\log\left(\frac{L}{\pi a}\sin\frac{\pi\ell}{L}\right) + U = S_B$$

Open systems: same formulae with different $\tilde{U}$

$$\frac{\tilde{U} - U}{2} = \log g$$

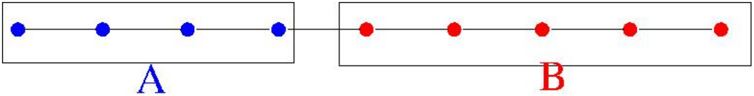
g =boundary entropy [Affleck, Ludwig 1991]

- Hamiltonian

$$H = \sum_{k=1}^{N} H_{k,k+1}$$

- Consider the ground state with $\rho = |0\rangle\langle 0|$
- Block of spins in the space interval $[1, \ell]$ is subsystem A
- The rest of the ground state is subsystem B

$\Longrightarrow$ Entanglement of a block of spins in the space interval $[1, \ell]$ with the rest of the ground state as a function of ℓ



E-Entropy and Universality

- Powers of ρ easily accessible in CFT (replica trick)

$$S_\alpha(\ell) = \frac{1}{1-\alpha} \mathrm{Tr}_A(\rho_A^\alpha)$$

- h = scaling dimension of the operator responsible for the correction [Calabrese, Cardy 2004-2010]

$$S_\alpha(\ell) = \frac{c}{6}\left(1-\frac{1}{\alpha}\right)\log \ell + \overbrace{c'_\alpha + \underbrace{b_\alpha(\ell)\ell^{-\frac{2h}{\alpha}} + \dots}}^{non-universal}$$

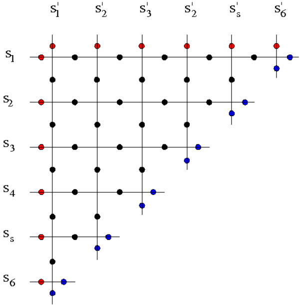
Conjectured

- Close to criticality $\xi \sim a^{-1}$ and $\ell \to \infty$

$$S_\alpha = \frac{c}{6}\left(1-\frac{1}{\alpha}\right)\log\frac{\xi}{a} + C'_\alpha + \overbrace{B_\alpha\left(\frac{\xi}{a}\right)^{-\frac{2x}{\alpha}} + \dots}$$

[Calabrese, Cardy, Peschel 2010]

- CTM is a very useful tool [Baxter (1972)]

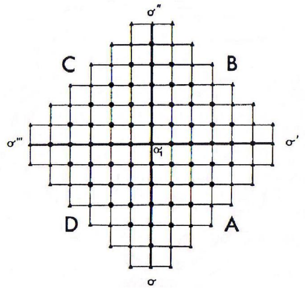


$$A_{\bar{s},\bar{s}'} = \sum_{\bullet} \prod w_i$$

- and analogously B, C, D with 90° rotations.

Partition function and CTM

- Now we can build up the whole lattice by using the 4 CTM's



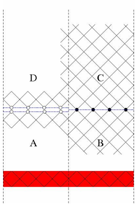
- Partition function

$$\mathcal{Z} = \sum_{\bar{\sigma},\bar{\sigma}',\bar{\sigma}'',\bar{\sigma}'''} A_{\bar{\sigma}\bar{\sigma}'} B_{\bar{\sigma}'\bar{\sigma}''} C_{\bar{\sigma}''\bar{\sigma}'''} D_{\bar{\sigma}'''\bar{\sigma}} = \mathrm{Tr}(ABCD)$$

Reduced density matrix and CTM

- Now suppose to divide the spins in two subsystems A: $\bar{\sigma}_A = (\sigma_1, ..., \sigma_p)$ and B: $\bar{\sigma}_B = (\sigma_{p+1}, ..., \sigma_L)$, i.e. $\bar{\sigma} = (\bar{\sigma}_A, \bar{\sigma}_B)$
- Reduced density matrix of subsystem A

$$\rho_A(\bar{\sigma}_A, \bar{\sigma}'_A) = \sum_{\bar{\sigma}_B} \langle \bar{\sigma}_A, \bar{\sigma}_B | 0 \rangle \langle 0 | \bar{\sigma}'_A, \bar{\sigma}_B \rangle = \mathrm{Tr}_B \langle \bar{\sigma}_A | 0 \rangle \langle 0 | \bar{\sigma}'_A \rangle$$



$$\rho_A = (ABCD)_{\bar{\sigma},\bar{\sigma}'} \quad \Longrightarrow \quad S_\alpha = \frac{1}{1-\alpha} \log \mathrm{Tr}_A \rho_A^\alpha$$

XYZ model

Hamiltonian

$$H_{XYZ} = -\sum_{k=1}^{N}(J_x\sigma_k^x\sigma_{k+1}^x + J_y\sigma_k^y\sigma_{k+1}^y + J_z\sigma_k^z\sigma_{k+1}^z)$$

- commutes with transfer matrix of 8-vertex model
- Useful parametrization

$$\Gamma = \frac{J_y}{J_x} = \frac{1 + k^2\mathrm{sn}^2(i\lambda;k)}{1 - k^2\mathrm{sn}^2(i\lambda;k)} \qquad \Delta = \frac{J_z}{J_x} = -\frac{\mathrm{cn}(i\lambda;k)\,\mathrm{dn}(i\lambda;k)}{1 - k^2\mathrm{sn}^2(i\lambda;k)}$$

$$0 < k < 1 \text{ and } 0 \leq \lambda \leq I(k')$$

Diagonalization of CTM

- One can prove that, in XYZ, $A = C$ and $B = D$.
- In the thermodynamic limit the following formula holds for the diagonalized CTM [Baxter (1977)]

$$\rho_A = ABCD = (AB)^2 = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & x^3 \end{pmatrix} \otimes \ldots$$

where

$$x = e^{-\frac{\pi\lambda}{I(k)}} = e^{-\epsilon}$$

$\epsilon = -\frac{\pi\lambda}{I(k)}$ depends on the XYZ parameters through elliptic functions and $I(k)$ is the elliptic integral of 1st kind of modulus k

- $\rho = e^{\epsilon \mathcal{H}_{CTM}}$ where $\mathcal{H}_{CTM}$ is a operator with integer spectrum

$$\mathcal{H}_{CTM} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \otimes \ldots$$

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Entanglement entropy of XYZ model

The trace of the reduced density matrix

$$\mathcal{Z} = \mathrm{Tr}\rho_A = \prod_{j=1}^{\infty}(1 + x^j) \qquad \text{and} \qquad S_A = -\epsilon\frac{\log \mathcal{Z}}{\partial\epsilon} + \log \mathcal{Z}$$

leads to the final formula for Von Neumann

$$S_A = \epsilon\sum_{j=1}^{\infty}\frac{j}{(1 + e^{j\epsilon})} + \sum_{j=1}^{\infty}\log(1 + e^{-j\epsilon})$$

and for Rényi entropy ($q = \sqrt{x} = nome$)

$$S_\alpha = \frac{\alpha}{\alpha - 1}\sum_{j=1}^{\infty}\log(1 + q^{2j}) + \frac{1}{1-\alpha}\sum_{j=1}^{\infty}\log(1 + q^{2j\alpha})$$

that can also be written in theta function terms

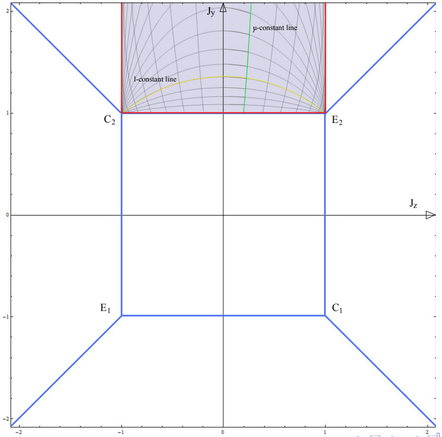
$$S_\alpha = \frac{1}{6(1-\alpha)}\left[\alpha\log\frac{\theta_4(0,q)\theta_3(0,q)}{\theta_2^2(0,q)} + \log\frac{\theta_2^2(0,q^\alpha)}{\theta_3(0,q^\alpha)\theta_4(0,q^\alpha)}\right]$$

Phase diagram of XYZ model

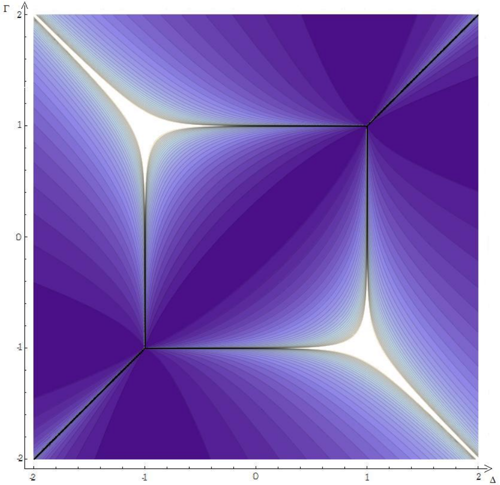
Approaching criticality the [Calabese - Cardy (2004)] formula holds

$$S_A = \frac{c}{6} \log \frac{\xi}{a} + \mathrm{cost.}$$

everywhere but at the $E_{1,2}$ points



Isoentropic lines



$$\mathcal{Z}(x) = \prod_{j=1}^{\infty}(1 + x^j) = x^{-\frac{1}{12}} \chi_{1,2}^{Ising}(i\epsilon/\pi)$$

and

$$\mathrm{Tr}\rho^\alpha = \frac{\chi_{1,2}^{Ising}(i\alpha\epsilon/\pi)}{\left[\chi_{1,2}^{Ising}(i\epsilon/\pi)\right]^\alpha}$$

Critical XXZ line: approached for $x \to 1$. Use modular transformation to get this limit $\tilde{x} = e^{-\frac{\pi^2}{\epsilon}}$

$$\mathrm{Tr}\rho^\alpha = 2^{\frac{\alpha-1}{2}} \frac{\chi_{1,1}^{Ising}(i\alpha\epsilon/\pi) - \chi_{2,1}^{Ising}(i\alpha\epsilon/\pi)}{\left[\chi_{1,1}^{Ising}(i\epsilon/\pi) - \chi_{2,1}^{Ising}(i\epsilon/\pi)\right]^\alpha}$$

Similar to [Calabrese, Cardy, Peschel (2011)] but with $c = \frac{1}{2}$ characters, not $c = 1$ (!): $\mathcal{H}_{CTM}$ is the hamiltonian of a free fermion $\longrightarrow$ property of many integrable models [Peschel (2010)]

The sine-Gordon limit

Scaling limit of the XYZ model:

- lattice spacing $a \to 0$ while
- mass gap kept constant

$$M = 8\pi \left(\frac{\sin \mu}{\mu} \right) \left(\frac{\ell}{4} \right)^{\pi/\mu} a^{-1}$$

where

$$\mu \equiv \pi \left(1 - \frac{\beta^2}{8\pi} \right) = \arccos(-J_z) \qquad , \qquad \ell^2 \equiv \frac{1 - J_y^2}{1 - J_z^2}$$

- $\Longrightarrow$ sine-Gordon theory [Johnson, Krinsky, McCoy 1973 - Luther 1974]
- J_z is connected to the parameter β of sine-Gordon
- J_y scales in the scaling limit $a \to 0 \quad \Longrightarrow \quad J_y \to 1^-$

Entanglement in bipartite SG

- Exact entanglement entropy of a bipartite XYZ model in the sine-Gordon limit

$$S_{sG} = -\frac{\pi}{12}\frac{\ln(1-k) - 3\ln 2}{\arctan\sqrt{\frac{1+J_z}{1-J_z}}} - \frac{\ln 2}{2} + O(1/\ln(a))$$

or, taking $J_z = -\cos\pi\left(1 - \frac{\beta^2}{8\pi}\right)$

$$S_{sG} = \frac{1}{6}\ln\left(\frac{1}{Ma}\right) + \frac{1}{6}\ln\left(\frac{\sin\left[\pi\left(1-\frac{\beta^2}{8\pi}\right)\right]}{\left(1-\frac{\beta^2}{8\pi}\right)}\right) + O(1/\ln(a))$$

Minimal models + ϕ_{13}

- Continuum limit of ABF models on square lattice. CTM diagonalization is given and the calculation of ρ_A straightforward [Franchini, De Luca (2012)]
- Can be generalized to FB non-unitary models [Bianchini, Ercolessi, FR, Pearce, work in progress]
- Renyi entropy

$$S_\alpha = \frac{1}{1-\alpha} \mathrm{Tr}_A \rho_A^\alpha = \frac{1}{1-\alpha} \log \mathcal{Z}_\alpha - \frac{\alpha}{1-\alpha} \log \mathcal{Z}_1$$

expression very complicated in terms of theta functions, but expanding:

$$S_\alpha = \frac{\alpha+1}{\alpha} \frac{c_{eff}}{12} \log \xi + U_\alpha + A_\alpha(\xi)$$

Open problems

- Taking $p, p' \to \infty$ in $\mathcal{M}_{p,p'}$ but with their ratio fixed, one accesses E-Entropy for log-CFT and their off-critical lattice realizations (percolations,...) [Pearce, Seaton (2011)]

- Is c_{eff} the correct leading coefficient predicition from CFT? c-theorem like arguments do not work for non-unitary cases. One has to compute 4-pt correlators with twist fields

$$\langle \Phi | \mathcal{T}\mathcal{T} | \Phi \rangle \qquad \text{instead of} \qquad \langle \mathcal{T}\mathcal{T} \rangle$$

  to compute ρ^α with replica trick. Progresses have been done [Sierra et al. (2013)]

- What is the interpretation of E-Entropy in non-unitary theories?

- *"The answer is yes, but... what was the question?"* [W. Allen]: We know the 2D classical lattice model, we can compute formally S_α, but what is the quantum Hamiltonian we are dealing with?

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Quantum critical hamiltonian

At criticality $\mathcal{U}_q(sl(2))$ invariant XXZ model [Alcaraz, Barber, Batchelor (1988) - Pasquier, Saleur (1990)]

$$H = -J \sum_{n=1}^{N-1} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \frac{q+q^{-1}}{2} \sigma_n^z \sigma_{n+1}^z) + \frac{q-q^{-1}}{2} (\sigma_1^z - \sigma_N^z)$$

Can be rewritten in terms of Temperley-Lieb operators

$$H = -J \sum_{n=1}^{N-1} e_n$$

$$e_n^2 = -(q+q^{-1}) e_n \quad , \quad e_n e_{n\pm 1} e_n = e_n \quad , \quad e_n e_m = e_m e_n \text{ if } |n-m| > 1$$

What happens off-criticality? Exercise done so far for the $r = 4$ case. Try to write the hamiltonian using tile operators F_n and exploit tile algebra to show that

$$H = -J \sum_{n=1}^{N-1} F_n$$

with

$$F_n F_{n+1} F_n = F_n \quad , \quad (F_{2n})^2 = \beta_1 F_{2n} \quad , \quad (F_{2n+1})^2 = \beta_2 F_{2n+1}$$

where $\beta_1 = \beta_2 = -(q + q^{-1})$ at the critical point $t = 0$.
It resembles TL-algebra, but with **two** parameters ($\Longrightarrow$ elliptic algebras?)

Summary

- Von Neumann and Rényi E-Entropies are crucial tools to study entanglement in quantum systems. In integrable models, they can be calculated using integrable techniques.
- Corner Transfer Matrix technique allows the exact calculation of bipartite E-Entropy in spin chains. Having the exact formula at hand, one can test some of the open issues about entanglement in these models.
- A suitable scaling limit yields the sine-Gordon model and E-entropy can be calculated for its bipartition. It is the first calculation of this kind for an interacting field theory.
- An integrable way to compute finite size E-Entropy is to be developed. It would complement the present knowledge by new precious information.

Conclusions

- Entanglement entropy is a new way to approach interesting problems in theoretical physics and it should be better understood in (integrable) QFT, as it seems crucial in the solution of challenging paradoxes, like the information loss in black holes.
- It also stimulates progresses in mathematics, in the best tradition of the integrability approach.

Thank you!!!