The Nambu-Goto string spectrum and the TBA

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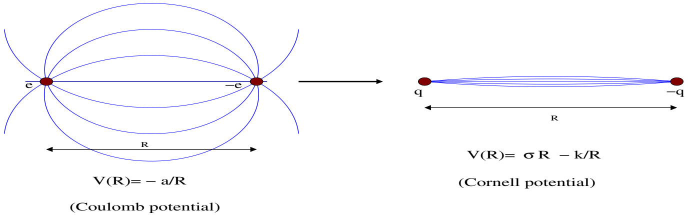
Mainly based on:

1. S. Dubovsky, R. Flauger, V. Gorbenko, "Solving the simplest theory of quantum gravity", [arXiv:1205.6805]
2. M. Caselle, D. Fioravanti, F. Gliozzi, R. Tateo, "Quantisation of the effective string with TBA", [arXiv:1305.1278]

• The present understanding of non perturbative QCD is mainly based on the idea that the confining regime of Yang-Mills theories is described by some kind of effective string model.

• A natural Ansatz for the interquark potential in 3+1D is the Coulomb/Linear (UV/IR) hybrid Cornell potential:



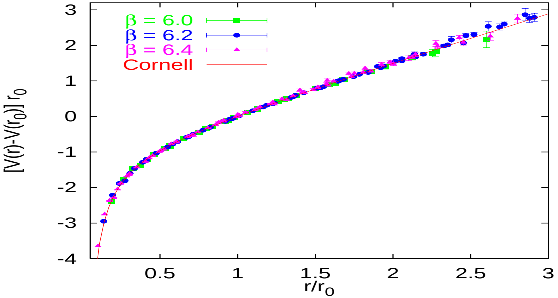
where σ is the string tension.

Coulombic part: one-gluon exchange?

Linear: Confinement part.

$\Longrightarrow$ Using lattice simulations of pure gauge theory one can easily check that $V(R)$ rises linearly and the flux lines are confined in a thin flux tube.

• Further, the Cornell's potential fits very well the numerical results:



(Bali hep-th/0001312: 4d, gauge group $SU(3)$, $\beta = 6/g_{YM}$.)

• However, an equally good numerical agreement is obtained in $2+1D$ where the Coulomb potential is logarithmic. In addition, the coefficient k appears to depend only on the dimensionality D of the spacetime!

On the Lattice:

- The Yang-Mills Euclidean action is:

$$S_{YM} = \frac{1}{4} \int d^D x \; F^i_{\mu\nu} F^{i\ \mu\nu},$$

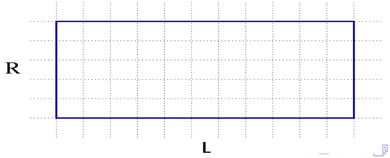
where $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g_{YM}\ f_{ijk} A^j_\mu A^k_\nu$ is the field strength tensor.

- No quarks! (Only the flux-tube properties are studied.)
- Discretisation on a D-dimensional hypercubic lattice, the vector potential is defined on the links: $A^i_\mu(x) \to A^i_\mu(n)$.
- Defining $U_\mu(n) \in \mathbf{G}$ as

$$U_\mu(n) = e^{\tau . A_\mu(n)},$$

  then the simplest gauge-invariant observable is the Wilson loop:

$$W(R,L) = \mathrm{Tr} \prod_{n_\mu \in \gamma_{(R,L)}} U_\mu(n).$$



At large L

$$\langle W(R,L)\rangle \sim e^{-L\ V(R)}.$$

- If we assume that $V(R)$ is dominated, for large R , by the linear term σR then we end up with the celebrated "area law" for the Wilson loop:

$$\langle W(R,L)\rangle \sim e^{-\sigma RL+p(R+L)+b} \ .$$

- The area term is responsible for confinement while the perimeter and the constant term b are non universal contributions depending on the discretization scheme.
- The field theory groundstate in presence of a confining or solitonic string, spontaneously breaks translational and rotational symmetries for a total of $3(D-2)$ generators:

$$SO(D) \longrightarrow SO(2) \times SO(D-2)$$

with the appearance of (only) $D-2$ Nambu-Goldstone modes or phonons.

- For spacetime-dependent symmetries there is no a one-to-one mapping between the number of broken generators and the number of Nambu-Goldstone bosons: local rotations and translations of the string are not all linearly independent.
- At quantum level, the flux-tube fluctuations can be described by a suitable 2d massless quantum field theory, where the bosonic fields describe the transverse displacements of the flux tube.
- Thus

$$\langle W(R,L)\rangle = e^{-\sigma RL + p(R+L)+b} Z(R,L),$$

where $Z(R,L)$ is the partition function of the massless theory in 2d.

- In the infrared limit $L \gg R \gg 1$, for any 2d massless QFT

$$\lim_{L\to\infty} \frac{1}{L} \log Z(R,L) = -E_0(R) \simeq \frac{\pi c_{\mathrm{IR}}}{24R}, \quad (R \gg 1)$$

where c_{IR} is the Casimir coefficient, or Virasoro central charge and E_0 is the ground state energy.

- For a fluctuating surface described by $D-2$ free boson fields $c_{\text{IR}} = D-2$ and

$$V(R) = -\lim_{L\to\infty} \frac{1}{L} \log\langle W(R,L)\rangle \sim \sigma R - \frac{(D-2)\pi}{24R}\ ,\ (R >> 0).$$

(Lüscher, Symanzik and Weisz.)

- The coefficient of $1/R$ can be measured with rather good precision and the results are in very good agreement with the LSW prediction.
- The corresponding action is

$$S = S_{cl} + S_0[X]\ ,\quad S_0[X] = \frac{\sigma}{2}\int d^2\xi\,(\partial_\alpha X\cdot\partial^\alpha X)\,.$$

- But there are observed deviations, and the first corrections compatible with the spacetime symmetries are

$$S = S_{cl}+S_0[X]+\sigma\int d^2\xi\left[c_2(\partial_\alpha X\cdot\partial^\alpha X)^2 + c_3(\partial_\alpha X\cdot\partial^\beta X)(\partial_\beta X\cdot\partial^\alpha X)\right]+S_b+.$$

where S_b is the boundary action characterizing the open string.

- The full Lorentz invariance of the target space, should be still respected nonlinearly by the expanded action. This gives

$$c_2 = \frac{1}{8}, \quad c_3 = -\frac{1}{4},$$

with this extra constraint we can write

$$S = S_{cl} + S_0[X] - \frac{1}{2\pi^2\sigma} \int d^2\xi \, T\bar{T} + S_b + \dots$$

where T and $\bar{T}$ are the holomophic and antiholomorphic components of the 2d stress-energy tensor with the standard CFT normalisation.

- $T\bar{T}$ is an "integrable" irrelevant perturbation appearing in many RG integrable flows connecting nontrivial UV and IR conformal field theory fixed points! (Zamolodchikov, Martins, Ravanini, Saleur, Fendley, DDT..)

- The action S reproduces the expansion of the Nambu-Goto string:

$$S_{NG} = \sigma \int dA = \sigma \int_0^L dl \int_0^R dr \sqrt{g},$$

where g is the determinant of the worldsheet metric :

$$g = \det(g_{\alpha\beta}) = \det\ \partial_\alpha X^\mu(l,r) \partial_\beta X^\mu(l,r).$$

- Thus S_{NG} measures the area swept out by the boson fields X^μ , i.e. in the physical (transverse) gauge:

$$X^1(l,r) = l, \quad X^2(l,r) = r.$$

- A naive quantisation (Arvis 1983) of the NG string leads to the spectrum

$$E_{(n,\bar{n})}(R) = \sqrt{\sigma^2 R^2 + 4\pi\sigma \left(n + \bar{n} - \frac{D-2}{12}\right) + \left(\frac{2\pi(n-\bar{n})}{R}\right)^2},$$

for periodic BCs in the R direction. The integers $n, \bar{n}$ define the total energy $2\pi n/R$ $(2\pi\bar{n}/R)$ of the left (right) moving phonons.

- Similarly, for the open string with fixed ends, where $n = \bar{n}$, one has

$$E_n(R) = \sqrt{\sigma^2 R^2 + 2\pi\sigma\left(n - \frac{D-2}{24}\right)},$$

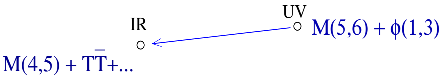
and the quark-antiquark potential corresponding to the NG model is:

$$V(R) = E_0(R) = \sigma R - \frac{(D-2)\pi}{24R} - \frac{(D-2)^2\pi^2}{1152\sigma R^3} + \dots$$

In conclusion:

- The effective string action is strongly constrained by Lorentz invariance.
- In the long string limit, the first few possible contributions are universal and coincide with those of the NG action.
- This fact explains why NG describes so well the infrared limit of Wilson loops or Polyakov correlators.

- Aliosha Zamolodchikov (1991) has proposed variants of the TBA and exact S-matrix approaches describing RG flows among pairs of CFTs.
- The simplest instance concerns the line of second order phase transitions connecting the tricritical Ising CFT to the Ising CFT:



- Massless particles confined on a infinite line or a ring naturally separate into right and left movers.
- Only one species of particles –the goldstino– is present.
- The right-right and left-left mover scattering is trivial, while the left-right scattering is described by the amplitude

$$S(p,q) = \frac{2\sigma + ipq}{2\sigma - ipq} \ ,$$

  where σ sets the scale, p is the momentum of the right mover and $-q$ the momentum of the left mover.
- In the limit $\sigma \to \infty$, $S(p,q) \to 1$, right and left movers decouple and the scale invariance of the model is fully restored at $\sigma = \infty$.

The corresponding TBA equations with periodic BCs are:

$$\epsilon(p) = Rp - \int_0^\infty \frac{dq}{2\pi}\phi(p,q)\,\bar{L}(q), \quad \bar{\epsilon}(p) = Rp \; - \int_0^\infty \frac{dq}{2\pi}\phi(p,q)\,L(q)\ ,$$

where $\epsilon(p)$ and $\bar{\epsilon}(p)$ are the pseudoenergies for the right and the left movers, respectively,

$$\phi(p,q) = -i\partial_q \log S(p,q)\ ,$$

and

$$L(p) = \log(1 + e^{-\epsilon(p)})\ ,\quad \bar{L}(p) = \log(1 + e^{-\bar{\epsilon}(p)})\ .$$

The groundstate energy is

$$E^{(\mathrm{TBA})}(\sigma, R) = -\frac{\pi}{6R}c(\sqrt{\sigma}R) = -\int_0^\infty \frac{dp}{2\pi}(L(p) + \bar{L}(p))\ ,$$

$c(R\sqrt{\sigma})$ is the (flowing) effective central charge with:

$$c_{\mathrm{UV}} = c_{\mathrm{TIM}} = c(0) = \frac{7}{10} \ , \ \ c_{\mathrm{IR}} = c_{\mathrm{IM}} = c(\infty) = \frac{1}{2} \ .$$

The energy levels obtained through the TBA method are automatically defined with respect to the vacuum energy in infinite space:

$$\lim_{R \to \infty} E_0^{(\mathrm{TBA})}(R) = 0 \ ,$$

a normalization that differs from the perturbative definition about the UV fixed point by a bulk contribution $F_0 R$:

$$E_0^{(\mathrm{TBA})}(R) = -\frac{\pi c_{\mathrm{UV}}}{6R} - F_0 R + \text{regular terms} \ , \ (R \simeq 0) \ .$$

(For the TIM $\rightarrow$ IM massless flow, $F_0 = 2\sigma$.)

The normalisation for the energy levels that matches the UV expansion is:

$$E_n(R) = E_n^{(\mathrm{TBA})}(R) + F_0 R \ .$$

The bulk term is the analogous of the linear term in the quark-antiquark potential, whose origin traces back to the classical contribution S_{cl} to the string action:

$$E_0(R) = F_0 R - \frac{\pi c_{\mathrm{IR}}}{6R} + \dots \ , \ (\sqrt{\sigma} R \gg 1) \ .$$

From the TBA equations one can obtain an exact asymptotic expansion for the scaling function $f^{(\mathrm{TBA})}(t) = \frac{1}{2\pi} R E_0^{(\mathrm{TBA})}(R)$:

$$\begin{aligned} f^{(\mathrm{TBA})}(t) = & -\frac{1}{24} - \frac{1}{48} t - \frac{1}{48} t^2 + \left(-\frac{5}{192} + \frac{49}{400}\pi^2 \right) t^3 \\ & + \left(-\frac{7}{192} + \frac{49}{100}\pi^2 \right) t^4 + \left(-\frac{7}{128} + \frac{441}{320}\pi^2 - \frac{2883}{245}\pi^4 \right) t^5 \\ & + \left(-\frac{11}{128} + \frac{539}{160}\pi^2 - \frac{723819}{9800}\pi^4 \right) t^6 + \dots \qquad (t = \pi/(12\sigma R^2)) \end{aligned}$$

The first three terms reproduce the large σ expansion of the effective action

$$S_\sigma = S_{\rm IM}^{\rm CFT} - \frac{1}{2\pi^2\sigma}\int d^2\xi\, T\bar{T} \ .$$

Indeed:

$$f^{(\rm pert)}(t) = -\frac{c_{\rm IR}}{12} + \left(\frac{c_{\rm IR}}{24}\right)^2 \alpha - \left(\frac{c_{\rm IR}}{24}\right)^3 \alpha^2 + O(\alpha^3)$$
$$= -\frac{1}{24} - \frac{1}{48}t - \frac{1}{48}t^2 + O(t^3) \ , \quad (\alpha = -48t) \ .$$

- The appearance in $f^{(\rm TBA)}(t)$ of coefficients with nonzero trascendentality at order $O(t^3)$ and greater are related to contributions of other irrelevant operators.
- The leading part S_1 of the Zamolodchikov's S matrix at large σ

$$S(p,q) = e^{ipq/\sigma - i(pq/\sigma)^3/12 + \dots} = S_1(p,q)e^{-i(pq/\sigma)^3/12+\dots}$$

  selects precisely the zero trascendentality terms in IR expansion which match the large R expansion of a NG type groundstate energy.
- To see this, we replace Zamolodchikov's kernel with

$$\phi(p,q) = -i\partial_q \log S_1(p,q) = p/\sigma \ .$$

The corresponding excited state TBA equations are:

$$\epsilon(p) = Rp - \frac{p}{\sigma}\int_{\bar{\mathcal{C}}}\frac{dq}{2\pi}\log_{\bar{\mathcal{C}}}(1+e^{-\bar{\epsilon}(q)}),$$

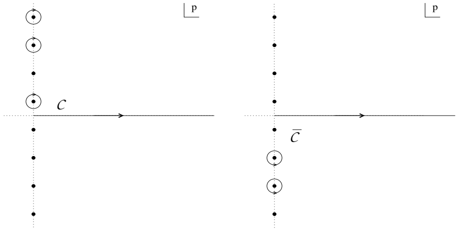
$$\bar{\epsilon}(q) = Rq - \frac{q}{\sigma}\int_{\mathcal{C}}\frac{dp}{2\pi}\log_{\mathcal{C}}(1+e^{-\epsilon(p)}),$$

with

$$E(R) = -\int_{\mathcal{C}}\frac{dp}{2\pi}\log_{\mathcal{C}}(1+e^{-\epsilon(p)}) - \int_{\bar{\mathcal{C}}}\frac{dq}{2\pi}\log_{\bar{\mathcal{C}}}(1+e^{-\bar{\epsilon}(q)}) + \sigma R,$$

where $\log_{\mathcal{C}}$ is the continuous branch logarithm, $\mathcal{C}$ and $\bar{\mathcal{C}}$ are integration contours running from 0 to ∞ on the real axis for the groundstate, but for excited states they circle around a finite number of poles $\{p_i\}$ and $\{\bar{p}_i\}$ of $\partial_p\log_{\mathcal{C}}(\dots)$ and $\partial_p\log_{\bar{\mathcal{C}}}(\dots)$:

$$\epsilon(p_j) = i\pi(2n_j+1)\ , \quad \bar{\epsilon}(\bar{p}_j) = -i\pi(2\bar{n}_j+1)\ , \quad (n_j, \bar{n}_j \in \mathbb{N})\ .$$



Setting

$$\epsilon(p) = Rp\kappa \ , \quad \bar{\epsilon}(p) = Rp\bar{\kappa} \ ,$$

we find the constraints

$$\kappa = 1 + \frac{4}{\pi\bar{\kappa}\sigma R^2}\mathrm{Li}_2(-1, \bar{\mathcal{C}}) \ , \quad \bar{\kappa} = 1 + \frac{4}{\pi\kappa\sigma R^2}\mathrm{Li}_2(-1, \mathcal{C}) \ .$$

$\mathrm{Li}_2(z, \mathcal{C})$ denotes the continuous branch dilogarithm:

$$\mathrm{Li}_2(z, \mathcal{C}) = -\int_{\mathcal{C}} \frac{dq}{2\pi} \log_{\mathcal{C}}(1 - ze^{-q}) = \mathrm{Li}_2(z) + 4\pi^2 m - i2\pi n \log(z) \ , \ (m, n \in \mathbb{N})$$

with $\mathrm{Li}_2(-1) = -\frac{\pi^2}{12}$.

A subsector of the spectrum corresponds to

$$\mathrm{Li}_2(-1,\mathcal{C}) = -\frac{\pi^2}{6}(c_{\mathrm{IR}} - 24n), \quad \mathrm{Li}_2(-1,\bar{\mathcal{C}}) = -\frac{\pi^2}{6}(c_{\mathrm{IR}} - 24\bar{n}),$$

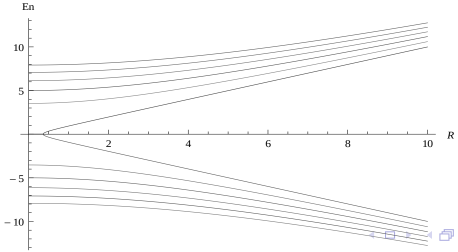
with $c_{\mathrm{IR}} = 1/2$ and

$$n = \sum_j n_j \ , \ \bar{n} = \sum_j \bar{n}_j .$$

The final result is:

$$E_{(n,\bar{n})}(R) = \sigma R(\kappa+\bar{\kappa}-1) = \sqrt{\sigma^2 R^2 + 4\pi\sigma\left(n+\bar{n}-\frac{c_{\mathrm{IR}}}{12}\right) + \left(\frac{2\pi(n-\bar{n})}{R}\right)^2} > 0,$$

and $\tilde{E}_{(n,\bar{n})}(R) = -E_{(n,\bar{n})}(R) < 0$ (??):



- In the large R limit, the result for the positive sector matches, up to $O(1/\sigma^3)$, the perturbative CFT calculations and generates only and all the zero trascendentality terms of the Zamolodchikov's $TIM \to IM$ flow.
- We may be tempted to discard the negative energy sector, but there are spectral singularities connecting the two branches. The most evident is the tachyonic critical point at

$$R_{cr} = \sqrt{\frac{\pi c_{IR}}{3\sigma}},$$

or, equivalently at the temperature $1/R_{cr}$.

The Hagedorn's temperature

- From the point of view of a QFT at finite temperature $T = 1/R$, this critical point is consequence of the exponential growth of the degeneracy of the energy levels at large energy E .
- In the current case, the degeneracy corresponds to the number of decompositions of n and $\bar{n}$ into distinct positive integers without regard to the order.
- This is the degeneracy of a free fermionic system on a circle. The generating function is

$$\sum_{n=0}^{\infty} \varphi(n) q^n = \prod_{n=1}^{\infty} (1 + q^n) \, .$$

- The asymptotic behaviour of the level degeneracy for large n and $\bar{n}$ is known to be

$$\rho(n) = \varphi(n)\varphi(\bar{n}) \simeq \varphi(n)^2 = \frac{1}{16\sqrt{3n^3}} e^{2\pi\sqrt{n/3}}.$$

For large $n \simeq \bar{n}$ the energy is $E(n) \simeq \sqrt{8\pi\sigma n}$, so the degeneracy $\rho(n)$ grows as

$$\rho(n) = 3\left(\frac{\pi T_H}{3E}\right)^3 e^{E/T_H} = \rho_I(E)\frac{dE}{dn},$$

where

$$T_H = \sqrt{\frac{3\sigma}{\pi c_{IR}}}$$

is the Hagedorn's temperature. Indeed, T_H coincides with the upper limit temperature of the system:

$$T(E) = 1/\partial_E \log\rho(E),$$

$$T_H = \sup(T(E)).$$

Comparing this result with the tachyonic singularity at R_{cr} we obtain:

$$R_{cr} = 1/T_H.$$

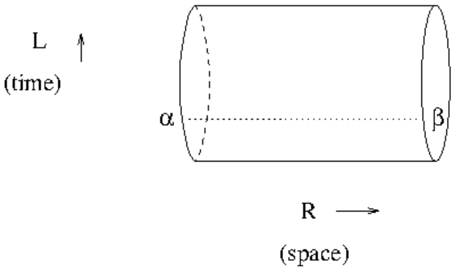
- These results can be generalised to many (possibly all) CFTs.
- With the same S-matrix, but using Bose statistics, one obtains the closed string NG spectrum with $D = 3 = c_{\mathrm{IR}} + 2$.
- In all cases the energy levels $E_{(n,\bar{n})}(R)$ are labeled by two integers n and $\bar{n}$ which depend on the monodromy of the dilogarithm in the complex plane.
- These energies can be parametrised in the form

$$E_{(n,\bar{n})}(R) = \sigma R + \mathcal{E} + \bar{\mathcal{E}},$$

where the two quantities $\mathcal{E}$ and $\bar{\mathcal{E}}$ obey the following consistency conditions:

$$\mathcal{E} = -\frac{\pi(c_{IR} - 24n)}{12(R + \bar{\mathcal{E}}/\sigma)} , \quad \bar{\mathcal{E}} = -\frac{\pi(c_{IR} - 24\bar{n})}{12(R + \mathcal{E}/\sigma)}.$$

- The solution of these two algebraic equations is exactly a NG-type spectrum.



Consider now the theory of a single Bose or Fermi field in 2d on a infinite strip of size R and left-right scattering amplitude

$$S_1(p,q) = e^{ipq/\sigma} \ .$$

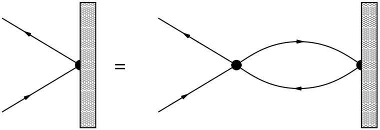
The relevant constraints for the boundary reflection factor $\mathcal{R}(p)$ are

$$\mathcal{R}(p)\mathcal{R}^*(p) = 1 \ ,$$

which coincides with unitary constraint, and

$$\mathcal{R}(p)\mathcal{R}(-p) = S_1(p,p) \ ,$$

corresponding to the equality between the following scattering diagrams (Ghoshal-Zamolodchikov):



Given the S-matrix, the minimal solution is

$$\mathcal{R}_0(p) = \sqrt{S_1(p,p)} = e^{ip^2/(2\sigma)} ,$$

while multi-parameter solutions have the general form

$$\mathcal{R}(p) = \mathcal{R}_0(p) \prod_{j=1}^{\infty} \mathcal{R}_{\delta_j}^{(2j-1)}(p) ,$$

with

$$\mathcal{R}_{\delta}^{(m)}(p) = e^{(ip)^m \delta} , \text{ (m odd)} ,$$

where $\mathcal{R}_{\delta}^{(m)}(p)\mathcal{R}_{\delta}^{(m)}(-p) = 1$.

The corresponding TBA equation is

$$\epsilon(p) = 2Rp + \Lambda(p) + \frac{p}{\sigma} \int_{\bar{\mathcal{C}}} \frac{dq}{2\pi} L(q) \ ,$$

where

$$\Lambda(p) = \log\left(\mathcal{R}_\alpha(-ip)/\mathcal{R}_\beta(ip)\right) \ ,$$

and

$$L(q) = \log_{\bar{\mathcal{C}}}\left(1 - e^{-\epsilon(q)}\right) \ , \quad (\text{Boson}, \ c_{IR} = 1)$$

or

$$L(q) = -\log_{\bar{\mathcal{C}}}\left(1 + e^{-\epsilon(q)}\right) \ , \quad (\text{Majorana Fermion}, \ c_{IR} = 1/2)$$

The vacuum and excited state energies are

$$E_n^{(\mathrm{TBA})}(R) = \int_{\bar{\mathcal{C}}} \frac{dp}{2\pi} L(p) \ .$$

For the "basic" boundary conditions

$$\Lambda(p) = \log\left(\mathcal{R}_0(-ip)/\mathcal{R}_0(ip)\right) = 0 \ ,$$

and the TBA equation reduces to

$$\epsilon(p) = 2Rp + \frac{p}{\sigma}\int_{\bar{\mathcal{C}}}\frac{dq}{2\pi}L(q) \ .$$

The following simple algebraic constraint is obtained:

$$E_n^{(\mathrm{TBA})}(R) = -\frac{\pi c(\bar{\mathcal{C}})}{12(2R + E_n^{(\mathrm{TBA})}(R)/\sigma)} \ ,$$

with $c(\bar{\mathcal{C}}) = c_{IR} - 24n$, ($n \in \mathbb{N}$). The positive energy solutions match the NG open string spectrum at $D = 3$ and $D = 5/2$:

$$E_n^{(\mathrm{TBA})}(R) + \sigma R = \sqrt{\sigma^2 R^2 + 2\pi\sigma\left(n - \frac{c_{IR}}{24}\right)} \ .$$

Next, consider the case

$$\mathcal{R}(p) = \mathcal{R}_0(p)\mathcal{R}^{(1)}_{\delta_1}(p) = e^{ip^2/2\sigma} e^{i\delta_1 p} \ ,$$

since $\Lambda(p) = \log\left(\mathcal{R}(-ip)/\mathcal{R}(ip)\right) = 2p\delta_1$, these boundary conditions correspond to a shift $R \to R + \delta_1$:

$$E_n^{(\mathrm{TBA})}(R, \delta_1) = E_n^{(\mathrm{TBA})}(R + \delta_1) \ .$$

For the Bosonic theory, this case corresponds to the b_1 term in the boundary action with $\delta_1 = -4b_1$:

$$S_b = \int d\xi_0 \left[b_1(\partial_1 X \cdot \partial_1 X) + b_2(\partial_1\partial_0 X \cdot \partial_1\partial_0 X) + b_3(\partial_1 X \cdot \partial_1 X)^2 + \dots \right] .$$

- This correction was calculated at first order in b_1 by Lüscher and Weisz using the $\zeta-$function and dimensional regularizations.
- They pointed out that this term corresponds to a shift in R and that this property extends to the next order in b_1: using the TBA approach we discovered that it is valid at any order of δ_1 .
- It is now established that Lorentz symmetry of the target space imposes $b_1 = 0$, yet the TBA approach is perfectly consistent.

Finally, consider the case

$$\mathcal{R}(p) = \mathcal{R}_0(p)\mathcal{R}^{(3)}_{\delta_2}(p) = e^{ip^2/(2\sigma)}e^{-ip^3\delta_2} \, , \;\; \Lambda(p) = \log\left(\mathcal{R}(-ip)/\mathcal{R}(ip)\right) = 2p^3\delta_2 \, .$$

The TBA equations are

$$\epsilon(p) = 2Rp + 2p^3\delta_2 + \frac{p}{\sigma}\int_{\bar{\mathcal{C}}}\frac{dq}{2\pi}L(q) \, .$$

The resulting modification to the NG spectrum at first order in δ_2 is:

$$E^{(\mathrm{TBA})}_{\{n_i\}}(R,\delta_2) = \sigma R(2\kappa - 2) = E^{(\mathrm{TBA})}_n(R) + \frac{\delta_2\pi^3}{4R^4}\left(\frac{1}{60} + 4\sum_i (n_i)^3\right) + \dots$$

with $n_i = 0, 1, \dots, \; n = \sum_i n_i$.

This equation matches the results obtained by Aharony and Klinghoffer with $\delta_2 = -4b_2$.

From lattice Montecarlo:

- $SU(2)$ lattice gauge theory in 3d:

$$(\sigma)^{3/2} b_2 \simeq -0.015(6), \quad \text{(Brandt 2011)}$$

- Ising gauge model in 3d:

$$(\sigma)^{3/2} b_2 \simeq 0.032(2), \quad \text{(Billò, Caselle, Gliozzi, Maineri, Pellegrini 2012)}$$

Using a trick introduced by Luscher one can get the exact partition function (rectangular Wilson loop) for an arbitrary CFT with equal conformal BCs on the four sides:

$$Z(R,T) = Z(T,R) = \sqrt{\sigma RT} \left(\frac{R}{T}\right)^{c/8} \int_0^\infty \frac{dt}{t^{c_{IR}/8+1}} e^{-\frac{1}{4t}-(E_0 T)^2 t} \left(q^{-1/24}\eta(q)\right)^{-c_{IR}/2}$$

(Universality: Kleban, Vassileva, D. Zagier.) At large R:

$$\ln Z(R,T) \sim -\sigma RT + \frac{c}{4} \ln R + \dots$$

The cusp contribution $\leftrightarrow$ Cardy-Peschel corner contribution

Conclusions

- Starting from a free boson theory one can recover the result by Arvis.
  Dubovsky-Flauger-Gorbenko (closed string)
  Caselle-Gliozzi-Fioravanti-RT (open string)
- The Nambu-Goto like spectrum can be obtained from any CFT both for open and periodic boundary conditions.
- Filling the negative energy Dirac sea? (We did not succeed and, after all, the TBA should already give the exact QFT result!)
- Is the theory incomplete or this "ultraviolet fragility" is an important aspect of some QFT signaling the emergence of new physics? (Dubovsky-Gorbenko-Mirbabayi)